Order Statistics

Def. Given distinct RVs \( Y_1, Y_2, \ldots, Y_n \), put

\[
Y_{(1)} := \min \{ Y_1, \ldots, Y_n \}
\]

\[
Y_{(2)} := \min \{ Y_k : Y_k > Y_{(1)} \}
\]

\[
Y_{(m)} := \min \{ Y_k : Y_k > Y_{(m-1)} \}, \quad m = 2, \ldots, n
\]

Of course, \( Y_{(n)} = \max \{ Y_1, \ldots, Y_n \} \). \( Y_{(m)} \) is called the \( m \)-th order statistic.

Example. If \( Y_1, Y_2, \ldots, Y_n \) are iid with density \( f(y) \), show that the joint pdf of \( V_i = Y_{(i)}, \ldots, V_n = Y_{(n)} \) is

\[
n! \prod_{k=1}^{n} f(Y_k), \quad v_1 < v_2 < \cdots < v_n.
\]

Solution. To simplify the notation, write \( \{ V \leq v \} \) for \( \{ Y_1 \leq v_1, \ldots, Y_n \leq v_n \} \). Next, observe that there are \( n! \) disjoint events of the form

\[
A_i := \{ Y_1 < Y_2 < \cdots < Y_n \}
\]

\[
A_i := \{ Y_2 < Y_1 < Y_3 < \cdots < Y_n \}
\]

\[
\vdots
\]

\[
A_{n!} := \{ Y_n < Y_{n-1} < \cdots < Y_1 \}
\]

and their union has probability one. Hence,

\[
\{ V \leq v \} = \bigcup_{i=1}^{n!} \{ V \leq v \} \cap A_i
\]

Furthermore, on \( A_i \), \( V_1 = Y_1, \ldots, V_n = Y_n \), on \( A_{i+1} \), \( V_1 = Y_1, V_2 = Y_2, \ldots, V_{n-1} = Y_{n-1}, \) and so on.
Thus,
\[
\{\forall \in A \wedge A_i = \{Y_i \leq v_i, \ldots, Y_n \leq v_n\} \wedge A_n, \\
\{\forall \in A \wedge A_2 = \{Y_2 \leq v_2, Y_1 \leq v_1, Y_3 \leq v_3; \ldots, Y_n \leq v_n\} \wedge A_n, \\
\text{and so on. Now, because the } Y_i \text{ are i.i.d., each of these intersections has the same probability},
\]

which is
\[
\int_{-\infty}^{v_1} \cdots \int_{-\infty}^{v_n} \mathcal{I}_{B_1}(y_1, \ldots, y_n) f(y_1) \cdots f(y_n) \\
dy_1 \cdots dy_n,
\]

where
\[
B_1 = \{(y_1, \ldots, y_n); y_1 < y_2 < \cdots < y_n\}.
\]

Taking the nth-order mixed partial derivative of the above integral yields \(f(y_1) \cdots f(y_n)\) if \(v_1 < \cdots < v_n\).

Then
\[
f_Y(y) = \frac{\partial^n}{\partial y_1 \cdots \partial y_n} \mathcal{I}(v \leq y) \\
= \frac{2^n}{\partial y_1 \cdots \partial y_n} \sum_{k=1}^{n} \mathcal{I}(v \leq y, A_k) \\
= n! \prod_{k=1}^{n} f(y_k), \text{ if } v_1 < \cdots < v_n.
\]

**Example.** We can now see that for an inhomogeneous Poisson process, \(f_{t_1, \ldots, t_n}(t_1, \ldots, t_n \mid n)\) is the same as the joint pdf of the order statistics of \(n\) i.i.d. RVs with pdf \(\frac{\lambda(x)}{\int_0^r \lambda(x) \, dx}\).
Observation. Suppose \( g(t_1, \ldots, t_n) = g(t_{\pi(1)}, \ldots, t_{\pi(n)}) \) for all permutations \( \pi \) of \( \{1, \ldots, n\} \). That is, \( g(t_1, \ldots, t_n) = g(t_{\pi(1)}, t_{\pi(2)}, t_{\pi(3)}, \ldots, t_{\pi(n)}) \) etc., for any permutation or rearrangement of \( t_1, \ldots, t_n \). Then

\[
E[g(T_1, \ldots, T_n) \mid N_t = n] = E[g(Y_1, \ldots, Y_n)],
\]

where \( Y_1, \ldots, Y_n \) are i.i.d. with pdf

\[
\frac{h(t)}{\int h(t) \, dt}, \quad t \geq 0.
\]

The most common example of such a function \( g \) is \( g(t_1, \ldots, t_n) = \sum_{k=1}^{n} h(t_k) \) for some function \( h(t) \).