Order Statistics

Def. Given distinct RVs $Y_1, Y_2, \dots, Y_n$, put

$$Y_{(1)} := \min\{Y_1, \dots, Y_n\}$$
$$Y_{(2)} := \min\{Y_k : Y_k > Y_{(1)}\}$$
$$\vdots$$
$$Y_{(m)} := \min\{Y_k : Y_k > Y_{(m-1)}\}, \quad m = 3, \dots, n$$

Of course, $Y_{(n)} = \max\{Y_1, \dots, Y_n\}$. $Y_{(m)}$ is called the mth order statistic.

Example. If $Y_1, Y_2, \dots, Y_n$ are iid with density $f(y)$, show that the joint pdf of $V_1 = Y_{(1)}, \dots, V_n = Y_{(n)}$ is

$$n! \prod_{k=1}^{n} f(v_k), \quad v_1 < v_2 < \cdots < v_n.$$

Solution. To simplify the notation, write $\{\underline{V} \le \underline{v}\}$ for $\{V_1 \le v_1, \dots, V_n \le v_n\}$. Next, observe that there are $n!$ disjoint events of the form

$$A_1 := \{Y_1 < Y_2 < \cdots < Y_n\}$$
$$A_2 := \{Y_2 < Y_1 < Y_3 < \cdots < Y_n\}$$
$$\vdots$$
$$A_{n!} := \{Y_n < Y_{n-1} < \cdots < Y_1\}$$

and their union has probability one. Hence,

$$\{\underline{V} \le \underline{v}\} = \bigcup_{i=1}^{n!} \{\underline{V} \le \underline{v}\} \cap A_i$$

Furthermore, on A_1, $V_1 = Y_1, \dots, V_n = Y_n$, on A_2, $V_1 = Y_2, V_2 = Y_1, V_3 = Y_3, \dots, V_n = Y_n$, and so on.

Thus,

$$\{\underline{V} \le \underline{v}\} \cap A_1 = \{Y_1 \le v_1, \ldots, Y_n \le v_n\} \cap A_1$$

$$\{\underline{V} \le \underline{v}\} \cap A_2 = \{Y_2 \le v_1, Y_1 \le v_2, Y_3 \le v_3, \ldots, Y_n \le v_n\} \cap A_2$$

and so on. Now, because the Y_k are iid, each of these intersections has the same probability, which is

$$\int_{-\infty}^{v_n} \cdots \int_{-\infty}^{v_1} I_{B_1}(y_1, \ldots, y_n)\, f(y_1) \cdots f(y_n)\, dy_1 \cdots dy_n,$$

where

$$B_1 := \{(y_1, \ldots, y_n) : y_1 < y_2 < \cdots < y_n\}$$

Taking the nth-order mixed partial derivative of the above integral yields $f(v_1) \cdots f(v_n)$ if $v_1 < \cdots < v_n$. Then

$$f_{\underline{V}}(\underline{v}) = \frac{\partial^n}{\partial v_1 \cdots \partial v_n} P(\underline{V} \le \underline{v})$$

$$= \frac{\partial^n}{\partial v_1 \cdots \partial v_n} \sum_{i=1}^{n!} P(\underline{V} \le \underline{v}, A_i)$$

$$= n! \prod_{k=1}^{n} f(v_k), \quad \text{if } v_1 < \cdots < v_n.$$

Example. We can now see that for an inhomogeneous Poisson process, $f_{T_1, \ldots, T_n | N_t}(t_1, \ldots, t_n | n)$ is the same as the joint pdf of the order statistics of n i.i.d. RVs with pdf $\dfrac{\lambda(\tau)}{\int_0^t \lambda(\tau)\, d\tau}$.

__Observation.__ Suppose $g(t_1,\dots,t_n) = g(t_{\pi_1},\dots,t_{\pi_n})$ for all permutations π of $\{1,\dots,n\}$. That is, $g(t_1,\dots,t_n) = g(t_2,t_1,t_5,t_3,t_4,t_6,\dots,t_n)$ etc. for any permutation or rearrangement of $t_1,\dots,t_n$. Then

$$E[g(T_1,\dots,T_n) \mid N_t = n] = E[g(Y_1,\dots,Y_n)],$$

where $Y_1,\dots,Y_n$ are i.i.d. with pdf $\dfrac{\lambda(\tau)}{\int_0^t \lambda(\tau)\,d\tau}$, $0 \le \tau \le t$.

The most common example of such a function g is $g(t_1,\dots,t_n) = \sum_{k=1}^{n} h(t_k)$ for some function $h(t)$.