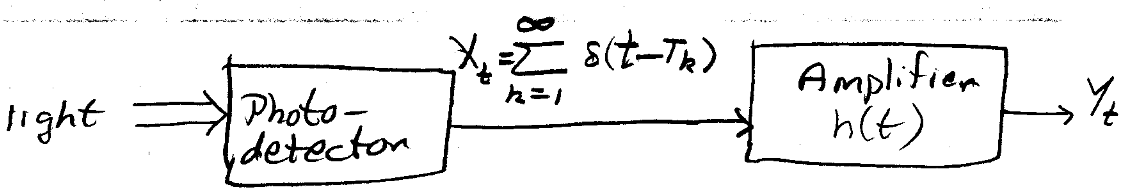
Shot Noise

In optical communications, light whose intensity varies with time according to $\lambda(\tau)$, $\tau \geq 0$, falls on a photodetector whose output drives an amplifier of impulse response $h(t)$. The photodetector generates photoelectrons according to a Poisson process of rate $\lambda(\tau)$.
Each electron acts like a unit impulse on the amplifier. Let $T_1 < T_2 < \cdots$ be the arrival times of the Poisson process.



$$
\begin{aligned}
Y_t &= \int_{-\infty}^{\infty} h(t-\tau) X_\tau \, d\tau \\
&= \int_{-\infty}^{\infty} h(t-\tau) \sum_{k=1}^{\infty} \delta(\tau - T_k) \, d\tau \\
&= \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} h(t-\tau) \delta(\tau - T_k) \, d\tau \\
&= \sum_{k=1}^{\infty} h(t - T_k).
\end{aligned}
$$

We call Y_t a shot-noise process
For fixed t, put $g(\tau) := h(t-\tau)$. Then

$$Z := Y_t = \sum_{k=1}^{\infty} g(T_k).$$

We call Z a shot-noise RV.

Let us compute $E[Z]$, assuming $g(\tau)=0$ for $\tau>t$. (This is true for causal $h(t)$). Then

$$Z = \sum_{k=1}^{N_t} g(T_k)$$

and we can use the law of total probability to write

$$E[Z] = \sum_{n=0}^{\infty} E[Z \mid N_t=n]\, P(N_t=n)$$

Now, $E[Z \mid N_t=n] = E\left[\sum_{k=1}^{N_t} g(T_k) \mid N_t=n\right]$

$$= E\left[\sum_{k=1}^{n} g(T_k) \mid N_t=n\right], \text{ by substitution}$$

Let $V_1, \ldots, V_n$ be iid with density $\lambda(\tau)/\int_0^t \lambda(\tau)d\tau$. Then the joint pdf of $V_{(1)}, \ldots, V_{(n)}$ is the same as the conditional joint pdf of $T_1, \ldots, T_n$ given $N_t=n$. So,

$$E[Z \mid N_t=n] = E\left[\sum_{k=1}^{n} g(V_{(k)})\right]$$

$$= E\left[\sum_{k=1}^{n} g(V_k)\right]$$

$$= \sum_{k=1}^{n} E[g(V_k)]$$

$$= n \underbrace{\int_0^t g(\tau)\lambda(\tau)d\tau / \Lambda([0,t])}_{=: C}$$

So,

$$E[Z] = \sum_{n=0}^{\infty} E[Z \mid N_t = n]\, P(N_t = n)$$

$$= \sum_{n=0}^{\infty} C n\, P(N_t = n)$$

$$= C \cdot E[N_t] = C \cdot \Lambda([0,t])$$

$$= \int_0^t g(\tau)\,\lambda(\tau)\,d\tau$$

If $g(\tau)$ is not zero for $\tau > t$, put
$\tilde{g}_t(\tau) := g(\tau)\, I_{[0,t]}(\tau)$ and

$$\tilde{Z}_t := \sum_{k=1}^{\infty} \tilde{g}_t(T_k)$$

If $\tilde{Z}_t \to Z$ as $t \to \infty$, then

$$E[Z] = \lim_{t\to\infty} E[\tilde{Z}_t] = \lim_{t\to\infty} \int_0^t \tilde{g}_t(\tau)\,\lambda(\tau)\,d\tau$$

$$= \lim_{t\to\infty} \int_0^t g(\tau)\, I_{[0,t]}(\tau)\,\lambda(\tau)\,d\tau$$

$$= \lim_{t\to\infty} \int_0^t g(\tau)\,\lambda(\tau)\,d\tau$$

$$= \int_0^{\infty} g(\tau)\,\lambda(\tau)\,d\tau.$$