Shot Noise

In optical communications, light whose intensity varies with time according to \( \lambda(t) \), \( t \geq 0 \), falls on a photodetector whose output drives an amplifier of impulse response \( h(t) \). The photodetector generates photoelectrons according to a Poisson process of rate \( \lambda(t) \). Each electron acts like a unit impulse on the amplifier. Let \( T_1 < T_2 < \cdots \) be the arrival times of the Poisson process.

\[
Y_t = \sum_{k=1}^{\infty} s(t-T_k)
\]

\[
Y_t = \int_{-\infty}^{\infty} h(t-z) \sum_{k=1}^{\infty} \delta(z-T_k) \, dz
\]

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= \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} h(t-z) \delta(z-T_k) \, dz
\]

\[
= \sum_{k=1}^{\infty} h(t-T_k).
\]

We call \( Y_t \) a shot-noise process.

For fixed \( t \), put \( g(z) := h(t-z) \). Then

\[
Z := Y_t = \sum_{k=1}^{\infty} g(T_k).
\]

We call \( Z \) a shot-noise RV.
Let us compute $E[Z]$, assuming $g(x) = 0$ for $x > t$. (This is true for causal $h(t)$). Then

$$Z = \sum_{k=1}^{N_t} g(T_k)$$

and we can use the law of total probability to write

$$E[Z] = \sum_{n=0}^{\infty} E[Z | N_t = n] P(N_t = n)$$

Now,

$$E[Z | N_t = n] = E\left[ \sum_{k=1}^{n} g(T_k) | N_t = n \right]$$

$$= E\left[ \sum_{k=1}^{n} g(T_k) | N_t = n \right], \text{ by substitution}$$

Let $V_1, \ldots, V_n$ be iid with density $\lambda(x)/\int_0^t \lambda(x) dx$. Then the joint pdf of $V_1, \ldots, V_n$ is the same as the conditional joint pdf of $T_1, \ldots, T_n$ given $N_t = n$. So,

$$E[Z | N_t = n] = E\left[ \sum_{k=1}^{n} g(V_k) \right]$$

$$= E\left[ \sum_{k=1}^{n} g(V_k) \right]$$

$$= \sum_{k=1}^{n} E[g(V_k)]$$

$$= n \int_0^t g(x) \lambda(x) dx \lambda([0,t])$$

$$=: c$$
So,

\[ E[Z] = \sum_{n=0}^{\infty} E[Z \mid N_t = n] \mathcal{P}(N_t = n) \]

\[ = \sum_{n=0}^{\infty} \mathcal{C} n \mathcal{P}(N_t = n) \]

\[ = \mathcal{C} \cdot E[N_t] = \mathcal{C} \cdot \Lambda([0,t]) \]

\[ = \int_{0}^{t} \mathcal{g}(z) \lambda(z) \, dz \]

If \( \mathcal{g}(z) \) is not zero for \( z > t \), put

\( \hat{\mathcal{g}}_{t}(z) := \mathcal{g}(z) \mathcal{I}_{[0,t]}(z) \) and

\[ \hat{Z}_t := \sum_{k=1}^{\infty} \hat{\mathcal{g}}_{t}(T_k) \]

If \( \hat{Z}_t \to Z \) as \( t \to \infty \), then

\[ E[Z] = \lim_{t \to \infty} E[\hat{Z}_t] = \lim_{t \to \infty} \int_{0}^{t} \hat{\mathcal{g}}_{t}(z) \lambda(z) \, dz \]

\[ = \lim_{t \to \infty} \int_{0}^{t} \mathcal{g}(z) \mathcal{I}_{[0,t]}(z) \lambda(z) \, dz \]

\[ = \lim_{t \to \infty} \int_{0}^{t} \mathcal{g}(z) \lambda(z) \, dz \]

\[ = \int_{0}^{\infty} \mathcal{g}(z) \lambda(z) \, dz. \]