Gaussian Processes

Def. A collection of RVs $\{X_\alpha\}$ is said to be **Gaussian** if every finite linear combination,

$$\sum_{i=1}^{n} c_i X_{\alpha_i}$$

is a scalar Gaussian RV. We consider constants to be Gaussian RVs with zero variance.

If $X = [X_1, \ldots, X_n]'$ is a random vector, then

$$m := E[X] := [E[X_1] \cdots E[X_n]]'$$ is its mean

vector, and

$$C := E[(X-m)(X-m)'] = [\,E[(X_i-m_i)(X_j-m_j)]\,]$$

is its covariance matrix.

Def. If $X = [X_1, \ldots, X_n]'$ is Gaussian with mean m and covariance C, we write $X \sim N(m, C)$.

Prop. If $X \sim N(m, C)$ and $Y := AX + b$ for some matrix A and column vector b, then

$$Y \sim N(Am+b,\ ACA')$$

Prop If $X \sim N(m, C)$, then the characteristic function of X is

$$\varphi_X(\nu) := E[e^{j\nu' X}] = e^{+j\nu' m - \nu' C \nu/2}$$

Proof. Since $X \sim N(m, C)$, $Y := \nu' X \sim N(\nu' m, \nu' C \nu)$ and so

$$E[e^{j\nu' X}] = E[e^{jY}] = E[e^{j\eta Y}]\Big|_{\eta=1}$$

$$= e^{j\eta m_Y - \eta \sigma_Y^2 \eta/2}\Big|_{\eta=1}$$

$$= e^{j\nu' m - \nu' C \nu/2} \qquad \blacksquare$$

Corollary If $X_1, \ldots, X_n$ are Gaussian & uncorrelated, i.e.,

$$E[(X_i - m_i)(X_j - m_j)] = \sigma_i^2 \delta_{ij},$$

then $X_1, \ldots, X_n$ are independent.

Proof. $\varphi_X(\nu) = e^{j\nu' m - \nu' C \nu/2}$

$$= e^{j\sum \nu_i m_i - \sum \sigma_i^2 \nu_i^2/2}$$

$$= \prod_{i=1}^{n} \underbrace{e^{j\nu_i m_i - \sigma_i^2 \nu_i^2/2}}_{N(m_i, \sigma_i^2)}$$

$$= \prod_{i=1}^{n} \varphi_{X_i}(\nu) \qquad \blacksquare$$

Prop If $Y \sim N(m, C)$ and C is invertible, then Y has a joint density,

$$f_Y(y) = \frac{\exp[-\frac{1}{2}(y-m)' C^{-1}(y-m)]}{(2\pi)^{n/2}\sqrt{\det C}}.$$

Proof, Recall that $y = G(x)$ where G is invertible with inverse $H := G^{-1}$ so that $x = H(y)$, then

$$f_Y(y) = f_X(H(y)) \, |\det dH(y)|$$

where

$$dH(y) = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \cdots & \frac{\partial h_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial h_n}{\partial y_1} & \cdots & \frac{\partial h_n}{\partial y_n} \end{bmatrix}.$$

Put $X := C^{-1/2}(Y-m)$ so that $H(y) = C^{-1/2}(y-m)$ and $dH(y) = C^{-1/2}$. Next, we have to find $f_X(x)$. Observe that X is zero mean and

$$E[XX'] = E[C^{-1/2}(Y-m)(Y-m)' C^{-1/2}]$$
$$= C^{-1/2} C C^{-1/2} = I$$

So, the components of X are uncorrelated. Since Y is Gaussian, X is Gaussian. So The components of X are $\perp\!\!\!\perp$ $N(0,1)$. Then

$$f_X(x) = \prod_{k=1}^{n} \frac{e^{-x_k^2/2}}{\sqrt{2\pi}} = \frac{e^{-x'x/2}}{(2\pi)^{n/2}},$$

We can now write,

$$f_Y(y) = \frac{e^{-[C^{-1/2}(y-m)]'[C^{-1/2}(y-m)]/2}}{(2\pi)^{n/2}} \underbrace{|\det C^{-1/2}|}_{=|(\det C)^{-1/2}|}$$

$\square$

<u>The complex case.</u> A complex RV $Z = X + jY$ is said to be continuous if X and Y are jointly continuous with joint density $f_{XY}(x,y)$.
We sometimes write

$$f_Z(z) = f_Z(x+jy) := f_{XY}(x,y).$$

<u>Example</u> $X, Y \perp\!\!\!\perp N(0, 1/2) \Rightarrow$

$$f_{XY}(x,y) = \frac{e^{-x^2}}{\sqrt{2\pi}\sqrt{1/2}} \cdot \frac{e^{-y^2}}{\sqrt{2\pi}\sqrt{1/2}} = \frac{e^{-|z|^2}}{\pi}$$

<u>Def</u> $\mathrm{var}(Z) := E[(Z - E[Z])(Z - E[Z])^*]$.
$$= \mathrm{var}(X) + \mathrm{var}(Y).$$

<u>Example</u>, If $Z = X + jY$ with $X, Y \perp\!\!\!\perp N(0, 1/2)$, then $\mathrm{var}(Z) = 1$.

<u>Def</u> $Z = X + jY$ is a complex random vector of dim. n if X and Y are real random vectors of dim. n. Also,

$$K := \mathrm{cov}(Z) := E[(Z - E[Z])(Z - E[Z])^H].$$

$$K = (C_X + C_Y) + j(C_{YX} - C_{XY})$$

where $C_X := \text{cov}(X)$, $C_Y = \text{cov}(Y)$, and

$$C_{XY} := \text{cov}(X,Y) := E[(X - m_X)(Y - m_Y)'].$$

If the $2n$-dim vector $[X_1, \ldots, X_n, Y_1, \ldots, Y_n]'$ has joint density f_{XY}, we write

$$f_Z(z) := f_{XY}(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

Def. We say $Z = X + jY$ is Gaussian if the $2n$-dim vector $[X', Y']'$ is Gaussian.

Def. A complex random vector $Z = X + jY$ is said to be circularly symmetric or proper if

$$C_X = C_Y \quad \text{AND} \quad C_{XY} = -C_{YX}.$$

Theorem. If an n-dim. complex Gaussian random vector $Z = X + jY$ is circularly symmetric with zero mean and covariance matrix K, then the chf of Z is

$$\mathbf{E}[e^{j(\nu' X + \theta' Y)}] = e^{-w^H K w/4}, \quad w := \nu + j\theta,$$

and if K is invertible,

$$f_{XY}(x,y) = \frac{e^{-z^H K^{-1} z}}{\pi^n \det K}, \quad z := x + jy.$$

Proof. Omitted.

<u>Fact.</u> If X_t is a Gaussian process; i.e., a family of Gaussian RVs, then

$$\int_a^b g(\tau) X_\tau \, d\tau$$

is a Gaussian RV. To see why this should be true, recall that this integral is defined as the limit in mean square of partial sums of the form

$$\sum_{i=1}^n g(\tau_i) X_{\tau_i} (\tau_i - \tau_{i-1}).$$

Since this is a linear combination of Gaussian RVs, the sum is Gaussian. Finally, the mean-square limit of Gaussian RVs is also Gaussian.

Using the above fact, it is easy to show that

$$Y_t = \int_a^b g(t,\tau) X_\tau \, d\tau \qquad (\#)$$

is a Gaussian process.

<u>Def.</u> The <u>correlation function</u> of a random process X_t is defined by

$$R(t,\tau) := E[X_t X_\tau^*].$$

Its <u>covariance function</u> is $C(t,\tau) := E[(X_t - m_t)(X_\tau - m_\tau)^*]$, where $m_t := E[X_t]$. If $m_t \equiv 0$, $C(t,\tau) = R(t,\tau)$.

If Y_t is given by (#),

$$R_{XY}(t,\tau) := E[X_t Y_\tau^*]$$

$$= E\left[X_t \int_a^b g(\tau,\theta)^* X_\theta^* \, d\theta\right]$$

$$= \int_a^b g(\tau,\theta)^* E[X_t X_\theta^*] \, d\theta$$

$$= \int_a^b g(\tau,\theta)^* R_X(t,\theta) \, d\theta \qquad (\#\#)$$

<u>Def.</u> We say X_t is <u>white noise</u> if $E[X_t]=0$ and $R_X(t,\theta) = \sigma^2 \delta(t-\theta)$.

For white noise, (##) becomes

$$R_{XY}(t,\tau) = \sigma^2 \int_a^b g(\tau,\theta) \delta(t-\theta) \, d\theta$$

$$= \begin{cases} \sigma^2 g(\tau,t)^*, & t \in [a,b], \\ 0, & t \notin [a,b]. \end{cases}$$

In general,

$$R_Y(t,\tau) = E[Y_t Y_\tau^*] = E\left[\left(\int_a^b g(t,s) X_s \, ds\right) Y_\tau^*\right]$$

$$= \int_a^b g(t,s) E[X_s Y_\tau^*] \, ds$$

$$= \int_a^b g(t,s) R_{XY}(s,\tau) \, ds$$

$$= \int_a^b g(t,s) \left(\int_a^b g(\tau,\theta)^* R_X(s,\theta)\right) d\theta \, ds.$$

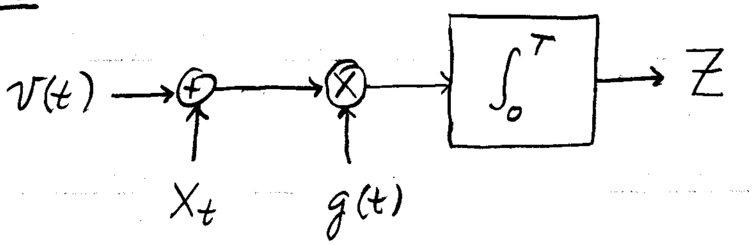
If X_t is white noise,

$$R_Y(t,\tau) = \sigma^2 \int_a^b g(t,s) g(\tau,s)^* \, ds.$$

In particular, if $Y = \int_a^b g(\tau) X_\tau \, d\tau$,

$$E[|Y|^2] = \sigma^2 \int_a^b |g(s)|^2 \, ds.$$

<u>Application.</u>



$$Z = \int_0^T g(t)[v(t) + X_t] \, dt = m + Y,$$

where $m := \int_0^T g(t) v(t) \, dt$ and $Y := \int_0^T g(t) X_t \, dt$.

If X_t is <u>Gaussian</u> white noise, then $Y \sim N(0, \sigma^2 \|g\|^2)$,

where $\|g\|^2 := \int_0^T |g(t)|^2 \, dt$. If $v(t) \equiv 0$, then

$$P(Z > \|g\|^2/2) = Q\left(\frac{\|g\|^2/2}{\sigma \|g\|}\right) = Q\left(\frac{\|g\|}{2\sigma}\right)$$

$$= Q\left(\sqrt{\frac{\|g\|^2}{4\sigma^2}}\right),$$

where $Q(x) := \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}} \, dt$.

Craig's Formula

Let $X \sim N(0,1)$ and put

$$Q(x_0) := \mathbb{P}(X > x_0) = \int_{x_0}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$$

Then

$$Q(x_0) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(\frac{-x_0^2}{2\cos^2\theta}\right) d\theta, \quad x_0 \geq 0, \; (*)$$

$$= \frac{1}{\pi}\int_0^{\pi/2} \exp\left(\frac{-x_0^2}{2\sin^2 t}\right) dt, \quad x_0 \geq 0. \; (**)$$

Eq $(**)$ is Craig's formula. To derive $(*)$, write

$$\mathbb{P}(X > x_0) = \mathbb{P}(X > x_0, Y \in \mathbb{R}) = \mathbb{P}((X,Y) \in D)$$

where $D := \{(x,y) : x > x_0\}$, Evaluate if $X \perp\!\!\!\perp Y$ with $Y \sim N(0,1)$.

Application. Let $V \geq 0$ be a RV with MGF $M_V(s) = E[e^{sV}]$. Then

$$E[Q(\sqrt{V})] = E\left[\frac{1}{\pi}\int_0^{\pi/2} \exp\left(\frac{-V}{2\sin^2 t}\right) dt\right]$$

$$= \frac{1}{\pi}\int_0^{\pi/2} E[e^{sV}]\,dt, \quad s = \frac{-1}{2\sin^2 t}$$

$$= \frac{1}{\pi}\int_0^{\pi/2} M_V\left(\frac{-1}{2\sin^2 t}\right) dt$$

$$= \frac{1}{\pi}\int_0^{1} M_V\left(\frac{-1}{2\theta^2}\right) \frac{d\theta}{\sqrt{1-\theta^2}}$$

$$= \frac{1}{2\pi}\int_{-1}^{1} M_V\left(\frac{-1}{2\theta^2}\right) \frac{d\theta}{\sqrt{1-\theta^2}}$$

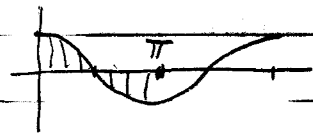
We can approximate this last integral using Chebyshev-Gauss quadrature. Thus,

$$E[Q(\sqrt{V})] \approx \frac{1}{2\pi}\sum_{k=1}^{n} w_k M_V\left(\frac{-1}{2\theta_k^2}\right), \qquad (*)$$

where $\theta_k := \cos\left(\frac{2k-1}{2n}\pi\right)$, $k=1,\dots,n$,

and

$$w_k = \pi/n, \quad k=1,\dots,n.$$



Now, observe that $\theta_k = 0 \iff \frac{2k-1}{n}\cdot\frac{\pi}{2} = \text{odd}\cdot\frac{\pi}{2}$, which is impossible if n is even. So for even n, we do not have to worry about $\lim_{s\downarrow-\infty} M_V(s)$.

Using the trig. identity $\cos(A-B) = \cos A\cos B + \sin A\sin B$, we see that

$$\theta_{n-\ell} = -\theta_{\ell+1} \text{ for } \ell = 0,\dots,n-1.$$

If n is even,

$$\theta_n = -\theta_1$$
$$\theta_{n-1} = -\theta_2$$
$$\vdots$$
$$\theta_{n-[\frac{n}{2}-1]} = -\theta_{\frac{n}{2}},$$

If n is odd

$$\theta_n = -\theta_1$$
$$\theta_{n-1} = -\theta_2$$
$$\vdots$$
$$\theta_{n-\frac{n-1}{2}} = -\theta_{\frac{n+1}{2}} = 0,$$

and $(*)$ becomes for even n

$$\frac{1}{n}\sum_{k=1}^{n/2} M_V\left(\frac{-1}{2\theta_k^2}\right).$$