The Probability of Error for
Binary Signaling over Multipath Channels

When the waveform \( \hat{s}(t) \) is transmitted over a multipath channel with delays \( T_k \) and gains \( G_k \), the response of the channel is

\[
\hat{s}(t) := \sum_{k} G_k \hat{s}(t-T_k)
\]

If a receiver attempts to measure this during the time interval \([0, T]\), the received waveform is*

\[
\rho(t) = \hat{s}(t) + W_t, \quad 0 \leq t \leq T,
\]

where \( W_t \) is AWGN with correlation function \( \sigma^2 \delta(t-\tau) \).

If the delays and gains are known at the receiver, and binary signaling is used, say waveforms \( \hat{s}_i(t) \), \( i=0,1 \), then the prob. of error is

\[
Q\left( \sqrt{\frac{2\sigma^2}{\gamma}} \right),
\]

where

\[
\gamma := \int_0^T \left| \hat{s}_i(t) - \hat{s}_0(t) \right|^2 dt = \int_0^T \left| \sum_k G_k \{ s_i(t-T_k) - s_0(t-T_k) \} \right|^2 dt
\]

We now assume that the \( s_i \) are short-duration pulses on \([0, t_p]\), where \( t_p \ll T \). Hence,

*We assume \( s(t) \), \( G_k \), and \( W_t \) are real.
\[ \bar{R}_n(t) := \bar{R}_n(t) - \bar{R}_0(t) \]

is zero for \( t \not\in [0, tp] \). We can then write

\[
\begin{align*}
\hat{d}^2 &= \int_0^T \left| \sum_k G_k \bar{R}_n(t-T_k) \right|^2 dt \\
&= \sum_k \sum_l G_k G_l \int_0^T \bar{R}_n(t-T_k) \bar{R}_n(t-T_l) dt \\
&= \sum_k \sum_l G_k G_l I_{[0,tp]}(T_k-T_l) \int_0^T \bar{R}_n(t-T_k) \bar{R}_n(t-T_l) dt.
\end{align*}
\]

If \( tp \) is small enough that the probability of \( |T_k-T_l| \leq tp \) is negligible for all \( k \neq l \), then

\[
\hat{d}^2 \approx \sum_k G_k^2 \int_0^T |\bar{R}_n(t-T_k)|^2 dt \\
&= \sum_k G_k I_{[0,T]}(T_k) \int_0^{\min(T_k+tp,T)} |\bar{R}_n(t-T_k)|^2 dt \\
&= \bar{\Phi}
\]

When \( T_k \leq T-tp \), this last integral is \( \int_0^{tp} |\bar{R}_n(t)|^2 dt =: \bar{d}^2 \), which is the square of the distance between \( \bar{R}_n \) and \( \bar{R}_0 \). If the prob. that \( T_k \in [T-tp, T] \) is negligible, then

\[
\hat{d}^2 \approx \bar{\Phi} \cdot \bar{d}^2,
\]

and the prob. of error is approximately

*Or if \( G_k \) is negligible for \( T_k > T-tp \), or if \( T=\infty \).
\( Q \left( \sqrt{\frac{d^2 \Phi}{4 \sigma^2}} \right) \)

The average bit-error prob. is approximately

\[
E \left[ Q \left( \sqrt{\frac{d^2 \Phi}{4 \sigma^2}} \right) \right].
\]

Recall Craig's formula,

\[
Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp \left[ -\frac{x^2}{2 \sin^2 t} \right] dt
\]

\[
= \frac{1}{\pi} \int_0^1 \frac{\exp \left[ -\frac{x^2}{2 y^2} \right]}{\sqrt{1-y^2}} dy
\]

\[
= \frac{1}{2\pi} \int_{-1}^1 \frac{\exp(-x^2/(2y^2))}{\sqrt{1-y^2}} dy.
\]

Put

\[
P_T(\eta) := E \left[ Q \left( \sqrt{2 \eta^2} \right) \right]
\]

\[
= \frac{1}{2\pi} \int_{-1}^1 \frac{E \left[ e^{-2 \eta^2/(2y^2)} \right]}{\sqrt{1-y^2}} dy
\]

\[
= \frac{1}{2\pi} \int_{-1}^1 \frac{M(-\eta/y^2)}{\sqrt{1-y^2}} dy,
\]

where \( M(\theta) := E[e^{\theta \Phi}], \ \theta \leq 0 \). Then the average BEP is approximately \( P_T(d^2/(8\sigma^2)) \).
Now, we can approximate \( \mathbb{P}(\eta) \) using Chebyshev-Gauss quadrature to write

\[
\mathbb{P}_e(\eta) \approx \frac{1}{K} \sum_{k=1}^{K/2} M\left(-\eta/y_k^2\right),
\]

where \( K \) is even and \( y_k := \cos\left(\frac{2k-1}{2K} \pi\right) \), \( k = 1, \ldots, K \).

**Example.** If \( G_k = 1 \) for all \( k \) and the \( T_k \) are the arrival times of a Poisson process, then

\[
\bar{\tau} = \sum_k I_{[0,T]}(T_k) = N_T \sim \text{Poisson}
\]

with parameter \( \Lambda([0,T]) = \int_0^T \lambda(t) \, dt \). Thus,

\[
M(\theta) = E\left[ e^{\theta N_T} \right] = e^{\Lambda([0,T])(e^\theta - 1)}.
\]

If the \( G_k \) are random, we need another approach.

*If \( \bar{\tau} = N_T \), \( P_0(\eta) = \sum_{n=0}^{\infty} Q\left(\sqrt{2\eta}n\right) \frac{\Lambda([0,T])^n}{n!} e^{-\Lambda([0,T])} \)