The Probability of Error for Binary Signaling over Multipath Channels

When the waveform $\xi(t)$ is transmitted over a multipath channel with delays T_k and gains G_k, the response of the channel is

$$\hat{\xi}(t) := \sum_k G_k \xi(t - T_k)$$

If a receiver attempts to measure this during the time interval $[0,T]$, the received waveform is*

$$\rho(t) = \hat{\xi}(t) + W_t, \quad 0 \le t \le T,$$

where W_t is AWGN with correlation function $\sigma^2 \delta(t-\tau)$.

If the delays and gains are known at the receiver, and binary signaling is used, say waveforms $\xi_i(t)$, $i=0,1$, then the prob. of error is

$$Q\left(\sqrt{\frac{\hat{d}^2}{4\sigma^2}}\right),$$

where

$$\hat{d}^2 = \int_0^T |\hat{\xi}_1(t) - \hat{\xi}_0(t)|^2 \, dt = \int_0^T \Big| \sum_k G_k \{\xi_1(t-T_k) - \xi_0(t-T_k)\}\Big|^2 dt$$

We now assume that the ξ_i are short-duration pulses on $[0, t_p]$, where $t_p \ll T$. Hence,

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*We assume $\xi(t)$, G_k, and W_t are real.

$$\xi_\Delta(t) := \xi_1(t) - \xi_0(t)$$

is zero for $t \notin [0, t_p]$. We can then write

$$\hat{d}^2 = \int_0^T \left| \sum_k G_k \xi_\Delta(t - T_k) \right|^2 dt$$

$$= \sum_k \sum_\ell G_k G_\ell \int_0^T \xi_\Delta(t - T_k)\, \xi_\Delta(t - T_\ell)\, dt$$

$$= \sum_k \sum_\ell G_k G_\ell\, I_{[0,t_p]}(|T_\ell - T_k|) \int_0^T \xi_\Delta(t - T_k)\, \xi_\Delta(t - T_\ell)\, dt.$$

If t_p is small enough that the probability of $|T_\ell - T_k| \le t_p$ is negligible for all $\ell \neq k$, then

$$\hat{d}^2 \approx \sum_k G_k^2 \int_0^T |\xi_\Delta(t - T_k)|^2 dt$$

$$= \underbrace{\sum_k G_k^2\, I_{[0,T]}(T_k)}_{=:\, \Phi} \int_{T_k}^{\min(T_k + t_p, T)} |\xi_\Delta(t - T_k)|^2 dt.$$

When $T_k \le T - t_p$, this last integral is $\int_0^{t_p} |\xi_\Delta(\tau)|^2 d\tau =: d^2$, which is the square of the distance between ξ_1 and ξ_0. If the prob. that $T_k \in [T - t_p, T]$ is negligible*, then

$$\hat{d}^2 \approx \Phi \cdot d^2,$$

and the prob. of error is appoximately

*or if G_k is negligible for $T_k > T - t_p$; or if $T = \infty$.

$$Q\left(\sqrt{\frac{d^2\Phi}{4\sigma^2}}\right)$$

The average bit-error prob. is approximately

$$E\left[Q\left(\sqrt{\frac{d^2\Phi}{4\sigma^2}}\right)\right].$$

Recall Craig's formula,

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left[\frac{-x^2}{2\sin^2 t}\right] dt$$

$$= \frac{1}{\pi}\int_0^1 \frac{\exp\left[\frac{-x^2}{2y^2}\right]}{\sqrt{1-y^2}}\, dy$$

$$= \frac{1}{2\pi}\int_{-1}^1 \frac{\exp(-x^2/(2y^2))}{\sqrt{1-y^2}}\, dy.$$

Put

$$P_T(\eta) := E\left[Q\left(\sqrt{2\eta\Phi}\right)\right]$$

$$= \frac{1}{2\pi}\int_{-1}^1 \frac{E\left[e^{-2\eta\Phi/(2y^2)}\right]}{\sqrt{1-y^2}}\, dy$$

$$= \frac{1}{2\pi}\int_{-1}^1 \frac{M(-\eta/y^2)}{\sqrt{1-y^2}}\, dy,$$

where $M(\theta) := E[e^{\theta\Phi}]$, $\theta \le 0$. Then the average BEP is approximately $P_T(d^2/(8\sigma^2))$.

Now, we can approximate $P_T(\eta)$ using Chebyshev-Gauss quadrature to write

$$P_T(\eta) \approx \frac{1}{K} \sum_{k=1}^{K/2} M(-\eta/y_k^2),$$

where K is even and $y_k := \cos\left(\frac{2k-1}{2K}\pi\right)$, $k=1,\ldots,K$.

<u>Example</u>. If $G_k = 1$ for all k and the T_k are the arrival times of a Poisson process, then*

$$\Phi = \sum_k I_{[0,T]}(T_k) = N_T \sim \text{Poisson}$$

with parameter $\Lambda([0,T]) = \int_0^T \lambda(\tau)\,d\tau$. Thus,

$$M(\theta) = E[e^{\theta N_T}] = e^{\Lambda([0,T])(e^\theta - 1)}.$$

If the G_k are random, we need another approach.

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*If $\Phi = N_T$, $P_e(\eta) = \sum_{n=0}^{\infty} Q(\sqrt{2\eta n})\, \frac{\Lambda([0,T])^n}{n!} e^{-\Lambda([0,T])}$