Marked Temporal Poisson Processes

Let N_t be a Poisson process with intensity function $\lambda(t)$ and arrival times $T_1, T_2, \ldots$.

Given $T_1 = t_1, T_2 = t_2, \ldots$, let $G_1, G_2, \ldots$ be conditionally independent with G_k having conditional density $f_{t_k}(g)$, where for each t, $f_t(g)$ is a pdf in the variable g. We call the set of random points $\{(T_k, G_k)\}$ a <u>marked</u> Poisson process. The G_k are called <u>marks</u>.

Consider a function $h(t,g)$, and define the shot-noise RV

$$Y := \sum_{k=1}^{N_T} h(T_k, G_k).$$

We need the MGF of Y. To use the law of total probability, we begin with

$$E[e^{sY} \mid N_T = n, T_1 = t_1, \ldots, T_n = t_n]$$

$$= E\Big[\exp\Big(s \sum_{k=1}^{n} h(t_k, G_k)\Big) \Big| \cdots\Big]$$

$$= \prod_{k=1}^{n} E[e^{sh(t_k, G_k)} \mid \cdots]$$

$$= \prod_{k=1}^{n} \int e^{sh(t_k, g)} f_{t_k}(g)\, dg.$$

Let us temporarily introduce the notation

$$\beta(t) := \int e^{s\,h(t,g)} f_t(g)\, dg$$

and

$$\gamma(t) := \ln \beta(t)$$

so that $\beta(t) = e^{\gamma(t)}$. Then

$$E[e^{sY} \mid \cdots] = \prod_{k=1}^{n} \beta(t_k) = \prod_{k=1}^{n} e^{\gamma(t_k)}$$

$$= \exp\Big[\sum_{k=1}^{n} \gamma(t_k) \Big].$$

Then

$$E[e^{sY}] = E\big[\, E[e^{sY} \mid N_T, T_1, \ldots, T_{N_T}]\,\big]$$

$$= E\Big[\exp\Big(\sum_{k=1}^{N_T} \gamma(T_k) \Big) \Big]$$

$$= E[\, e^{\theta Z}\,]\Big|_{\theta=1}, \quad Z := \sum_{k=1}^{N_T} \gamma(T_k)$$

$$= \exp\Big[\int_0^T \{ e^{\theta\gamma(t)} - 1 \} \lambda(t)\, dt \Big]\Big|_{\theta=1}$$

$$= \exp\Big[\int_0^T \Big\{ \int e^{s\,h(t,g)} f_t(g)\, dg - 1 \Big\} \lambda(t)\, dt \Big] \quad (*)$$

$$= \exp\Big[\int_0^T \int \{ e^{s\,h(t,g)} - 1 \} f_t(g)\, \lambda(t)\, dg\, dt \Big]$$

Notice that the MGF of Y looks exactly like what we would expect if the points $\{(T_k, G_k)\}$ came from a two-dimensional Poisson process with intensity function $\tilde{\lambda}(t,g) = \lambda(t) f_t(g)$.

From ⊛ we see that an important quantity is

$$\int e^{sh(t,g)} f_t(g)\, dg.$$

Of particular interest to us is the case

$$h(t,g) = a(t)\, g^2, \qquad \text{(\#)}$$

where $a(t)$ is some time function. We then have

$$\int e^{sh(t,g)} f_t(g)\, dg = \int e^{sa(t)g^2} f_t(g)\, dg$$
$$= M_t(s a(t)),$$

where

$$M_t(\theta) := E_t[e^{\theta G^2}],$$

where G is a generic RV with density $f_t(g)$. Common choices for $f_t(g)$ include Rayleigh, Nakagami-m, Rice, and lognormal. We now make an important observation. If G is Rayleigh, then $G = \sqrt{X}$, where X is chi-squared two degrees of freedom ($X \sim \chi_2^2$). If G is Nakagami-m, then $G = \sqrt{X}$, where $X \sim \chi^2_{2m}$. If G is Rice, then $G = \sqrt{X}$, where $X \sim \chi_n^2(\mu^2)$, where μ^2 is the noncentrality parameter. Now, the χ^2 is a special case of the gamma, and of the noncentral χ^2, both of which have a MGF in closed form. If G^2 is a gamma RV with parameters p and β, then

$$M_t(\theta) = \left(\frac{\beta}{\beta - \theta}\right)^p, \quad \operatorname{Re}\theta < \beta,\ p > 0.$$

If $G^2 \sim \chi_k^2$ then $p = k/2$ and $\beta = 1/2$. If $G^2 \sim \chi_n^2(\mu^2)$, then

$$M_t(\theta) = \frac{\exp[\theta\mu^2/(1-2\theta)]}{(1-2s)^{n/2}}.$$

Recall that G is lognormal if it has the form $G = e^X$ where $X \sim N(m, \sigma^2)$. The dB spread of G, σ_{dB}, the standard deviation of

$$10 \log_{10} G = \frac{10}{\ln 10} X.$$

Thus, $\sigma_{dB} = \frac{10}{\ln 10}\sigma$.* Of course, $G^2 = e^{2X} = e^{\tilde{X}}$,

where $\tilde{X} \sim N(2m, 4\sigma^2)$. Thus, G^2 is also lognormal with dB mean $\frac{20}{\ln 10} m$ and dB spread $\frac{20}{\ln 10}\sigma$.

In this case, all we can do is write

$$M_t(\theta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[\theta e^{\sqrt{2}\tilde{\sigma}t + \tilde{m}}\right] e^{-t^2}\, dt, \qquad \theta \le 0,$$

which is in the form for Hermite–Gauss quadrature,

$$\int_{-\infty}^{\infty} g(t) e^{-t^2} dt \approx \sum_{k=1}^{K} w_k g(t_k).$$

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* The dB mean of G is $m_{dB} = E[10 \log_{10} G] = \frac{10}{\ln 10} m$.

If $a(t)$ in ⊕ is $I_{[0,T]}(t)$ and $\lambda(t) = \lambda$, then

$$E[e^{sY}] = \exp\left[\lambda\left\{ \int_0^T M_t(s)\,dt - 1 \right\}\right].$$

We now turn to the problem of approximating integrals of the form

$$\int_0^1 q(t)\,dt \approx \sum_{k=1}^{K} w_k\, q(t_k),$$

where the nodes t_k and the weights w_k are given by Legendre-Gauss quadrature on $[0,1]$.

<u>Quadrature in MATLAB.</u> Simple MATLAB functions are available to provide the nodes and weights for different Gaussian quadratures at

http://eceservo.ece.wisc.edu/~gubner/researchUWB.shtml

The usage is

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[t, w, n] = ...(K);
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where t is a column vector, w is a row vector, and usually $n = K$. The desired approximation of the integral is

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w*q(t);
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where q is a function you have written so that it can accept a vector argument. The dimensions of $q(t)$ should be the dimensions of t.

Now suppose you want to approximate an integral with a parameter, say

$$r(b) := \int_0^1 q(bt)\,dt \approx \sum_{k=1}^{K} w_k\, q(b t_k).$$

More generally, you want to plot $r(b)$ for $b = b_1, \ldots, b_L$; i.e., you need to compute

$$r(b_\ell) = \sum_{k=1}^{K} w_k\, q(b_\ell t_k), \quad \ell = 1, \ldots, L.$$

Put $b = [b_1, \ldots, b_L]$ and use the MATLAB expression

$$\underset{1\times K}{r = w} * q(\underbrace{\underset{K\times 1}{t} * \underset{1\times L}{b}}_{K\times L});$$

$$= [r_1, \ldots, r_L].$$

Now you can plot(b,r).

This requires q to accept a matrix input and return a matrix output. For example, the MATLAB function lincmb.m that you used in HW#2 converts the input t to a column vector for processing and generates output values as a column vector internally. Prior to returning, this column vector is converted to the dimensions of the input t.

Power Delay Profile

For a multipath channel whose response to a transmitted waveform $\xi(t)$ is

$$\sum_k G_k \, \xi(t-T_k),$$

the power delay profile (pdp) is defined as

$$p(\tau) := E\Big[\sum_k G_k^2 \, \delta(\tau - T_k) \Big].$$

However, because it is difficult to work with delta functions, we consider the power delay cumulative distribution function (pd-cdf)

$$P(t) := \int_0^t p(\tau)\, d\tau$$

$$= \int_0^t E\Big[\sum_k G_k^2 \delta(\tau - T_k) \Big] d\tau$$

$$= E\Big[\sum_k G_k^2 \int_0^t \delta(\tau - T_k)\, d\tau \Big]$$

$$= E\Big[\sum_k G_k^2 \, I_{[0,t]}(T_k) \Big].$$

Hence, a more appropriate definition of $p(\tau)$ is

$$p(\tau) := \frac{d}{dt} P(t).$$

Let us compute $P(t)$ for a marked Poisson process.

The easy way to do this is to regard the marked Poisson process as a two-dimensional Poisson process. Then

$$\Phi := \sum_h G_h^2 I_{[0,t]}(T_h)$$

is a shot-noise RV with

$$E[\Phi] = \iint g^2 I_{[0,t]}(\tau)\, \tilde{\lambda}(\tau, g)\, dg\, d\tau$$

$$= \int_0^t \lambda(\tau)\left[\int g^2 f_\tau(g)\, dg\right] d\tau.$$

If $\lambda(\tau) = \lambda$ and $f_\tau(g)$ has a second moment of the form $m_2(\tau) = e^{-\mu\tau}$, then

$$P(t) = \lambda \int_0^t e^{-\mu\tau}\, d\tau = \frac{\lambda}{\mu}(1 - e^{-\mu t}),\ t \geq 0.$$

and

$$p(t) = \lambda e^{-\mu t},\ t \geq 0$$

The <u>mean excess delay</u> is

$$\bar{D} := \int_0^\infty t\, p(t)\, dt \Big/ \int_{-\infty}^\infty p(t)\, dt.$$

The <u>mean square excess delay</u> is

$$\overline{D^2} := \int_0^\infty t^2 p(t)\, dt \Big/ \int_{-\infty}^\infty p(t)\, dt.$$

The delay spread is

$$s := \sqrt{\overline{D^2} - (\overline{D})^2},$$

which is the standard deviation of the power delay profile.

For the example above, the normalized power delay profile is an exponential pdf with parameter μ. Hence, $\overline{D} = 1/\mu$, $\overline{D^2} = 2/\mu^2$, and $s = 1/\mu$.