## Measures

<u>Def.</u>
We say that $\mu$ is a <u>measure</u> on a space $\mathbb{X}$ if

(i) $\mu(\emptyset) = 0$
(ii) $\mu(A) \geq 0$ for $A \subset \mathbb{X}$
(iii) If $A_1, A_2, \ldots$ are disjoint subsets of $\mathbb{X}$, then
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

<u>REMARKS.</u> In general $\mu(A)$ is not defined for all subsets $A$ of $\mathbb{X}$. We denote the collection of subsets $A$ for which $\mu(A)$ is defined by $\mathcal{Q}$. For the above definition to make sense, it is sufficient that $\mathcal{Q}$ have the following three properties:

(i) $\emptyset \in \mathcal{Q}$
(ii) $A \in \mathcal{Q} \Rightarrow A^c \in \mathcal{Q}$
(iii) $A_1, A_2, \ldots$ in $\mathcal{Q} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{Q}$.

Such a collection of subsets is called a $\sigma$-algebra or a $\sigma$-field.

<u>Example 1.</u> Fix any $x_0 \in \mathbb{X}$, and put
$$\mu(A) := I_A(x_0) = \begin{cases} 1, & x_0 \in A, \\ 0, & x_0 \notin A. \end{cases}$$

<u>Example 2</u>. If $\mu$ is a measure and $c \geq 0$, then $\nu(A) := c\mu(A)$ defines another measure.

<u>Example 3</u>. If $\mu_1, \mu_2, \ldots$ are measures, then so are

$$\mu(A) := \sum_{n=1}^{\infty} \mu_n(A).$$

and

$$\nu(A) := \sum_{n=1}^{\infty} c_n \mu_n(A),$$

assuming all $c_n \geq 0$.

<u>Proposition.</u> If $A \subset B$, $\mu(A) \leq \mu(B)$

<u>Proof.</u> Write $B = A \cup (B \setminus A)$. Since this is a disjoint union,

$$\begin{aligned}\mu(B) &= \mu(A) + \mu(B \setminus A) \\ &\geq \mu(A)\end{aligned}$$

$\blacksquare$

<u>Def.</u> If $\mu(X) < \infty$, $\mu$ is called a finite measure. If $\mu(X) = 1$, $\mu$ is called a probability measure.

<u>Def.</u> The pair $(X, \mathcal{A})$ is called a <u>measurable space</u>, and the triple $(X, \mathcal{A}, \mu)$ is called a <u>measure space</u>. If $\mu(X) = 1$, $(X, \mathcal{A}, \mu)$ is called a <u>probability space.</u>

In a probability space, $0 \leq \mu(A) \leq 1$ for all $A \in \mathcal{A}$.

<u>Example 4</u>. Let $x_1, x_2, \ldots$ be distinct points in $\mathbb{X}$, and let $p_1, p_2, \ldots$ be nonnegative numbers. Then

$$\mu(A) = \sum_{n=1}^{\infty} p_n I_A(x_n)$$

defines a measure. Also note that

$$\mu(\{x_k\}) = \sum_{n=1}^{\infty} p_n I_{\{x_k\}}(x_n) = p_k.$$

If $A$ contains $x_i$ and $x_j$ but no other $x_n$, then

$$\mu(A) = p_i + p_j$$

<u>Fact.</u> <u>Lebesgue measure</u> $\mathcal{L}$ on $\mathbb{R}^d$ is characterized as follows:

$d=1$: $\mathcal{L}([a,b]) = b-a$

$d=2$: $\mathcal{L}([a,b]\times[c,d]) = (b-a)(c-d)$

$d=3$: $\mathcal{L}([a,b]\times[c,d]\times[e,f]) = (b-a)(c-d)(e-f)$

$\ldots$

In other words, the Lebesgue measure of an interval is its length, the Lebesgue measure of a rectangle is its area, and the Lebesgue measure of rectangular parallelepiped is its volume, ....

Def. A counting measure is a measure that takes only nonneg. integer values (or $+\infty$).

Example 5. Fix any $x_1, x_2, \ldots$ in $\mathbb{X}$ and put.

$$\mu(A) := \sum_{k=1}^{\infty} I_A(x_k)$$

Then $\mu(A) = \# x_k \in A$.

Example 6. If $X_1, X_2, \ldots$ are $\mathbb{X}$-valued random variables, then

$$N(A) := \sum_{k=1}^{\infty} I_A(X_k)$$

is **a** random counting measure. Its mean measure is

$$\Lambda(A) := E[N(A)].$$

Example 7. (Poisson Counting Measure). We say $N$ is a Poisson counting measure if whenever $A_1, \ldots, A_m$ are disjoint, $N(A_1), \ldots, N(A_m)$ are independent ($\perp\!\!\!\perp$), and

$$P(N(A) = n) = \frac{\Lambda(A)^n e^{-\Lambda(A)}}{n!}, \quad n = 0, 1, \ldots$$

and $\Lambda$ is a measure with no point masses.