**Definition**

We say that $\mu$ is a measure on a space $\Omega$ if:

1. $\mu(\emptyset) = 0$
2. $\mu(A) \geq 0$ for $A \in \Omega$
3. If $A_1, A_2, \ldots$ are disjoint subsets of $\Omega$, then
   \[ \mu\left( \bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n). \]

**Remarks.** In general, $\mu(A)$ is not defined for all subsets $A$ of $\Omega$. We denote the collection of subsets $A$ for which $\mu(A)$ is defined by $\mathcal{A}$. For the above definition to make sense, it is sufficient that $\mathcal{A}$ have the following three properties:

1. $\emptyset \in \mathcal{A}$
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3. $A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Such a collection of subsets is called a $\sigma$-algebra or a $\sigma$-field.

**Example 1.** Fix any $x_0 \in \Omega$, and put

\[ \mu(A) := I_A(x_0) = \begin{cases} 1, & x_0 \in A, \\ 0, & x_0 \notin A. \end{cases} \]
Example 2. If $\mu$ is a measure and $c \geq 0$, then $\nu(A) := c \mu(A)$ defines another measure.

Example 3. If $\mu_1, \mu_2, \ldots$ are measures, then $\mu(A) := \sum_{n=1}^{\infty} \mu_n(A)$, and $\nu(A) := \sum_{n=1}^{\infty} c_n \mu_n(A)$, assuming all $c_n \geq 0$.

Proposition. If $A \subseteq B$, $\mu(A) \leq \mu(B)$

Proof. Write $B = A \cup (B \setminus A)$. Since this is a disjoint union,

$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$

Def. If $\mu(\mathcal{X}) < \infty$, $\mu$ is called a finite measure.
If $\mu(\mathcal{X}) = 1$, $\mu$ is called a probability measure.

Def. The pair $(\mathcal{X}, \mathcal{A})$ is called a measurable space, and the triple $(\mathcal{X}, \mathcal{A}, \mu)$ is called a measure space. If $\mu(\mathcal{X}) = 1$, $(\mathcal{X}, \mathcal{A}, \mu)$ is called a probability space.

In a probability space, $0 \leq \mu(A) \leq 1$ for all $A \in \mathcal{A}$. 
Example 4. Let $x_1, x_2, \ldots$ be distinct points in $\mathbb{I}$, and let $\rho_1, \rho_2, \ldots$ be nonnegative numbers. Then

$$
\mu(A) = \sum_{n=1}^{\infty} \rho_n I_{[x_n]}(x) = \rho(A)
$$

defines a measure. Also note that

$$
\mu([x_n]) = \sum_{n=1}^{\infty} \rho_n I_{[x_n]}(x) = \rho_n.
$$

If $A$ contains $x_i$ and $x_j$ but no other $x_n$, then

$$
\mu(A) = \rho_i + \rho_j.
$$

Fact. Lebesgue measure $\mathcal{L}$ on $\mathbb{R}^d$ is characterized as follows:

- $d=1$: $\mathcal{L}(a,b] = b-a$
- $d=2$: $\mathcal{L}(a,b] \times (c,d] = (b-a)(d-c)$
- $d=3$: $\mathcal{L}(a,b] \times (c,d] \times (e,f] = (b-a)(d-c)(e-f)$
  
  \ldots

In other words, the Lebesgue measure of an interval is its length, the Lebesgue measure of a rectangle is its area, and the Lebesgue measure of a rectangular parallelepiped is its volume.
Def. A counting measure is a measure that takes only integer values (or +∞).

Example 5. Fix any \( x_1, x_2, \ldots \) in \( \Sigma \) and put,
\[
\mu(A) := \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} I_A(x_k)
\]
Then \( \mu(A) = \# x_k \in A \).

Example 6. If \( X_1, X_2, \ldots \) are \( \Sigma \)-valued random variables, then
\[
N(A) := \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} I_A(X_k)
\]
is a random counting measure. Its mean measure is
\[
\Lambda(A) := E[N(A)].
\]

Example 7. (Poisson Counting Measure). We say \( N \) is a Poisson counting measure if whenever \( A_1, \ldots, A_m \) are disjoint, \( N(A_1), \ldots, N(A_m) \) are independent (\( \Pi \)), and
\[
P(N(A)=n) := \frac{\Lambda(A)^n e^{-\Lambda(A)}}{n!}, \quad n=0, 1, \ldots
\]
and \( \Lambda \) is a measure with no point masses.