The Lebesgue Integral

Let $f(x)$ be defined on a measure space $(X, \mathcal{A}, \mu)$. We say f is a <u>simple function</u> if

$$f(x) = \sum_{k=1}^{n} c_k I_{A_k}(x)$$

for finite numbers $c_1, \ldots, c_n$ and <u>measurable sets</u> $A_k \in \mathcal{A}$. For such functions with all $c_k \ge 0$, put

$$I(f) := \sum_{k=1}^{n} c_k \mu(A_k)$$

<u>Def.</u> For $f \ge 0$,

$$\int f \, d\mu := \sup \{ I(\varphi) : 0 \le \varphi \le f \text{ and } \varphi \text{ is simple} \}$$

<u>Remark</u> It is not too hard to show that if

$$\sum_{k=1}^{n} c_k I_{A_k}(x) = \sum_{l=1}^{m} d_l I_{B_l}(x) \quad \text{for all } x \in X, \text{ then}$$

$$\sum_{k=1}^{n} c_k \mu(A_k) = \sum_{l=1}^{m} d_l \mu(B_l).$$

For general f, put

$$f^+(x) := \begin{cases} f(x), & \text{if } f(x) \ge 0 \\ 0, & \text{o.w.} \end{cases}$$

$$f^-(x) := \begin{cases} -f(x), & \text{if } f(x) < 0 \\ 0, & \text{o.w.} \end{cases}$$

If either $\int f^+ < \infty$ or $\int f^- < \infty$, we put

$$\int f := \int f^+ - \int f^-;$$

Otherwise, $\int f$ is <u>not defined</u>.

<u>Fact.</u> For most functions on $\mathbb{R}^d$ that we run into,

$\int f \, d\mathcal{L}$ turns out to be the usual $\int f(x)\, dx$

that you know from calculus.

<u>Two Exceptions</u>

(i) $\displaystyle\int \frac{\sin x}{x} \, d\mathcal{L}(x)$ does not exist because $\displaystyle\int \left[\frac{\sin x}{x}\right]^+ d\mathcal{L}(x) = \int \left[\frac{\sin x}{x}\right]^- d\mathcal{L}(x) = +\infty$. Although

$$\lim_{r\to\infty} \int_{-r}^{r} \frac{\sin x}{x} \, d\mathcal{L}(x) = \underbrace{\int_{-\infty}^{\infty} \frac{\sin x}{x} \, dx}_{\substack{\text{Riemann} \\ \text{integral}}} = \pi$$

(ii) Let A denote the rational numbers in $[0,1]$. Then $\mathcal{L}(A) = 0$ & so $\int I_A(x) \, d\mathcal{L}(x) = \mathcal{L}(A) = 0$. But the upper & lower Riemann sums for $\int_0^1 I_A(x)\, dx$ are different (0 and 1).

Lebesgue Monotone Convergence Theorem

If $0 \le f_n(x) \nearrow f(x)$, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Lebesgue Dominated Convergence Theorem

If $f_n(x) \to f(x)$ and $|f_n(x)| \le g(x)$ and $\int g \, d\mu < \infty$, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Example If μ is a counting measure, say

$$\mu(A) = \sum_{i=1}^{\infty} I_A(x_i)$$

for given points x_k, then

$$\int g \, d\mu = \sum_{i=1}^{\infty} g(x_i)$$

Example If N is a random counting measure with mean measure Λ, then $\int g \, dN = \sum g(X_k)$ is called a shot-noise random variable. and

$$E\left[\int g \, dN\right] = \int g(x) \, d\Lambda(x).$$

Example If $\Lambda(A) = \int_A f \, d\mu$, then

$$\int g \, d\Lambda = \int g f \, d\Lambda.$$

<u>Example</u> If N is a Poisson counting measure with mean measure Λ, then the moment generating function of the shot-noise RV

$$Y := \int g\,dN = \sum_{i=1}^{\infty} g(X_i)$$

is

$$E[e^{sY}] = \exp\left[\int [e^{sg(x)} - 1]\, d\Lambda(x) \right].$$

We derive this for the case that g is simple, say

$$g(x) = \sum_{k=1}^{n} c_k I_{A_k}(x).$$

Without loss of generality, we may assume the A_k are disjoint. Then

$$Y = \int g\,dN = \sum_{k=1}^{n} c_k N(A_k)$$

where the $N(A_k)$ are independent. Then

$$E[e^{sY}] = E[e^{s\sum_{k=1}^{n} c_k N(A_k)}]$$

$$= \prod_{k=1}^{n} E[e^{sc_k N(A_k)}]$$

$$= \prod_{k=1}^{n} e^{\Lambda(A_k)(e^{sc_k} - 1)}$$

$$= \exp\left[\sum_{k=1}^{n} (e^{sc_k} - 1)\Lambda(A_k) \right]$$

$$= \exp\left[\int \left(e^{s g(x)} - 1\right) d\Lambda(x) \right],$$

where we have used the fact that for $x \notin \bigcup_{k=1}^{n} A_k$, $g(x)=0$, which implies $e^{sg(x)} - 1 = 0$ as well.

For arbitrary g, let φ_n be simple, $\varphi_n \to g$, and such that $\underbrace{\int \varphi_n \, dN}_{=: Y_n} \to \underbrace{\int g \, dN}_{=: Y}$. Then

$Y_n \to Y$ & so $e^{sY_n} \to e^{sY}$. Then

$$E[e^{sY}] \overset{\text{hopefully}}{=} \lim_{n\to\infty} E[e^{sY_n}]$$

$$= \lim_{n\to\infty} \exp\left[\int \left(e^{s\varphi_n(x)} - 1\right) d\Lambda(x) \right]$$

$$= \exp\left[\lim_{n\to\infty} \int \left(e^{s\varphi_n(x)} - 1\right) d\Lambda(x) \right]$$

$$\overset{\text{hopefully}}{=} \exp\left[\int \left(e^{sg(x)} - 1\right) d\Lambda(x) \right].$$

<u>Def.</u> If N is any random counting measure, then its <u>probability generating functional (pgfl)</u> is

$$\Psi_N(g) := E\left[e^{-\int g(x) N(dx)} \right], \quad g \ge 0.$$

<u>Theorem.</u> If $\Psi_N(g) = \exp\left[\int \{e^{-g(x)} - 1\} \Lambda(dx) \right]$, then N is a Poisson counting measure with mean measure Λ.

<u>Sketch of Proof.</u>

Given disjoint $A_1, \ldots, A_J$, we must show $N(A_1), \ldots, N(A_J)$ are $\perp\!\!\!\perp$ Poisson RVs with parameters $\Lambda(A_1), \ldots, \Lambda(A_J)$. We compute their joint probability generating function. For $0 < z_j \le 1$, put $c_j := -\ln z_j \ge 0$ so that

$$\begin{aligned} z_1^{N(A_1)} \cdots z_J^{N(A_J)} &= e^{-c_1 N(A_1)} \cdots e^{-c_J N(A_J)} \\ &= \exp\Big[-\sum_{j=1}^{J} c_j N(A_j)\Big] \\ &= \exp\Big[-\int g(x)\, N(dx)\Big], \end{aligned}$$

if $g(x) := \sum_{j=1}^{J} c_j I_{A_k}(x)$. Thus,

$$\begin{aligned} E\big[z^{N(A_1)} \cdots z^{N(A_J)}\big] &= E\Big[e^{-\int g(x) N(dx)}\Big] \\ &= \exp\Big[\int \{e^{-g(x)} - 1\}\Lambda(dx)\Big] \\ &= \exp\Big[\sum_{k=1}^{J} (e^{-c_k} - 1)\Lambda(A_k)\Big]. \\ &= \prod_{k=1}^{J} \underbrace{e^{\Lambda(A_k)(z_k - 1)}}_{\text{Poisson}(\Lambda(A_k))\ \text{pgf}} \end{aligned}$$

$\blacksquare$