Basic Combinatorics

Let Z be a finite set containing n elements. Then we say that the cardinality of Z is n, and we denote this by $|Z| = n$.

We denote by Z^k the n-fold Cartesian product of Z with itself. Thus, $\underline{z} = (z_1, \ldots, z_k) \in Z^k$ is a typical element of Z^k. Observe that $|Z^k| = |Z|^k$, which is n^k.

We denote by $Z^{(k)}$ the set of all $\underline{z} = (z_1, \ldots, z_k) \in Z^k$ with distinct entries; i.e., such that $z_i \neq z_j$ for $i \neq j$. To create an arbitrary $\underline{z}$, choose any $z_1 \in Z$. Then there are only $n-1$ possible choices for $z_2 \neq z_1$. For z_3 there are only $n-2$ possible choices such that $z_3 \neq z_2$ and $z_3 \neq z_1$. Hence, for $\underline{z}$, there are only $n \cdot (n-1) \cdots (n-[k-1]) = n!/(n-k)!$ possibilities. We conclude that $|Z^{(k)}| = n!/(n-k)!$

Let $Z_{n,k} := \{\text{all subsets } V \subset Z : |V| = k\}$. For any such subset $V = \{v_1, \ldots, v_k\}$, $(v_1, \ldots, v_k) \in Z^{(k)}$ because to say $|V| = k$ means there are k distinct members of V. Of course, there are $k!$ permutations (re-arrangements) of $(v_1, \ldots, v_k)$ that also belong to $Z^{(k)}$. Now observe that $|Z_{n,k}| \cdot k! = |Z^{(k)}| = n!/(n-k)!$, and so

$$|Z_{n,k}| = \frac{n!}{k!\,(n-k)!} =: \binom{n}{k}.$$

Our interest here is $\mathcal{Z}_{n,k}$ because each $V \in \mathcal{Z}_{n,k}$ is <u>one</u> way of selecting k <u>distinct</u> elements from Z <u>without regard to order</u>. Thus, $|\mathcal{Z}_{n,k}|$ is the <u>total</u> number of ways of choosing k out of n objects without regard to order. For this reason, the symbol $\binom{n}{k}$ is read "n choose k."

The problem of selecting k out of n objects without regard to order can be recast as the problem of partitioning Z into two disjoint subsets, one of size k and the other of size $n-k$. More generally, consider the problem of partitioning Z into $J+1$ disjoint subsets of sizes $\underbrace{m_0, \ldots, m_J,}_{\text{where } m_0+\cdots+m_J=n.}$ For set number 0, there are $\binom{n}{m_0}$ possibilities. For set number 1, there are $\binom{n-m_0}{m_1}$ possibilities. For set number 2, there are $\binom{n-m_0-m_1}{m_2}$ possibilities. Finally, for set number J, there are $\binom{n-(m_0+\cdots+m_{J-1})}{m_J}$ possibilities. Hence, the number of such partitions of Z is

$$\binom{n}{m_0}\binom{n-m_0}{m_1}\cdots\binom{n-(m_0+\cdots+m_{J-1})}{m_J}$$

$$= \frac{n!}{m_0!\,(n-m_0)!}\,\frac{(n-m_0)!}{m_1!\,([n-m_0]-m_1)!}\cdots\frac{[n-(m_0+\cdots+m_{J-1})]!}{m_J!\,\{[n-(m_0+\cdots+m_{J-1})]-m_J\}!}$$

$$= \frac{n!}{m_0!\,m_1!\cdots m_J!} =: \binom{n}{m_0,\ldots,m_J}.$$

<u>multinomial coefficient</u>