Construction of Poisson Processes

Let Λ have no point masses and satisfy $0<\Lambda(\mathbb{X})<\infty$.
Let $X_1, X_2, \ldots$ be iid with $P(X_k \in A) = \Lambda(A)/\Lambda(\mathbb{X})$.
Let M be independent of the $\{X_k\}_{k=1}^{\infty}$ and have a Poisson probability mass function with parameter $\Lambda(\mathbb{X})$.

<u>Remark:</u> Since Λ has no point masses, $X_i \neq X_j$ with probability one, and thus

$$|\{X_1, \ldots, X_n\}| = n \quad \text{with probability one.}$$

Define the random counting measure

$$N(A) := \begin{cases} \sum_{k=1}^{M} I_A(X_k), & M \geq 1, \\ 0, & M = 0. \end{cases}$$

Let us first compute, using the law of total probability,

$$\begin{aligned} P(N(A)=n) &= \sum_{m=n}^{\infty} P(N(A)=n \mid M=m)\, P(M=m) \\ &= \sum_{m=n}^{\infty} P\Big(\sum_{k=1}^{M} I_A(X_k)=n \,\Big|\, M=m\Big) P(M=m) \\ &= \sum_{m=n}^{\infty} P\Big(\sum_{k=1}^{m} I_A(X_k)=n \,\Big|\, M=m\Big) P(M=m) \end{aligned}$$

by substitution,

$$= \sum_{m=n}^{\infty} \mathbb{P}\Big(\sum_{k=1}^{m} I_A(X_k) = n\Big)\,\mathbb{P}(M=m), \quad \text{by } \perp\!\!\!\perp.$$

$$= \sum_{m=n}^{\infty} \underbrace{\mathbb{P}\big(n \text{ of } X_k \in A \;\&\; (m-n) \text{ of } X_k \notin A\big)}_{\binom{m}{n}\mathbb{P}(X_k \in A)^n\,\mathbb{P}(X_k \notin A)^{m-n}}\,\mathbb{P}(M=m)$$

$$= \sum_{m=n}^{\infty} \frac{m!}{n!\,(m-n)!}\left[\frac{\Lambda(A)}{\Lambda(\mathbb{X})}\right]^n\left[\frac{\Lambda(A^c)}{\Lambda(\mathbb{X})}\right]^{m-n}\cdot\frac{\Lambda(\mathbb{X})^m}{m!}e^{-\Lambda(\mathbb{X})}$$

$$= \sum_{m=n}^{\infty} \frac{\Lambda(A)^n}{n!}e^{-\Lambda(\mathbb{X})}\cdot\frac{\Lambda(A^c)^{m-n}}{(m-n)!}$$

$$= \frac{\Lambda(A)^n}{n!}e^{-\Lambda(\mathbb{X})}\cdot e^{+\Lambda(A^c)}$$

$$= \frac{\Lambda(A)^n}{n!}e^{-\Lambda(A)}.$$

It remains to show that for disjoint $A_1, \dots, A_J$, $N(A_1), \dots, N(A_J)$ are $\perp\!\!\!\perp$. We proceed as before:

Let $n := n_1 + \dots + n_J$.

$$\mathbb{P}(N(A_1)=n_1, \dots, N(A_J)=n_J) = \sum_{m=n}^{\infty}\mathbb{P}(N(A_1)=n_1,\dots,N(A_J)=n_J \mid M=m)\cdot\mathbb{P}(M=m)$$

$$= \sum_{m=n}^{\infty}\mathbb{P}\Big(n_1 \text{ of } X_1,\dots,X_m \in A_1, \dots, n_J \text{ of } X_1,\dots,X_m \in A_J, \;\&\; \underbrace{m-n}_{=:n_0} \text{ of } X_1,\dots,X_m \in \underbrace{\Big(\bigcup_{j=1}^{J} A_j\Big)^c}_{=:A_0}\Big)\cdot\mathbb{P}(M=m)$$

$$= \sum_{m=n}^{\infty}\binom{m}{n_0, n_1, \dots, n_J}\left[\frac{\Lambda(A_0)}{\Lambda(\mathbb{X})}\right]^{n_0}\cdots\left[\frac{\Lambda(A_J)}{\Lambda(\mathbb{X})}\right]^{n_J}\cdot\frac{\Lambda(\mathbb{X})^m e^{-\Lambda(\mathbb{X})}}{m!}$$

$$= \frac{\Lambda(A_1)^{n_1}\cdots\Lambda(A_J)^{n_J}}{n_1!\cdots n_J!}\, e^{-\Lambda(X)} \cdot \underbrace{\sum_{m=n}^{\infty}\frac{\Lambda(A_0)^{m-n}}{(m-n)!}}_{=e^{\Lambda(A_0)}}$$

$$= \prod_{j=1}^{J}\frac{\Lambda(A_j)^{n_j}}{n_j!}\, e^{-\Lambda(A_j)},$$

since $\Lambda(\mathbb{X}) - \Lambda(A_0) = \Lambda(A_0^c) = \Lambda\left(\bigcup_{j=1}^{J} A_j\right) = \sum_{j=1}^{J}\Lambda(A_j)$.

We thus see that N is a Poisson counting measure, or a <u>Poisson process</u>.

We next turn to the case $\Lambda(\mathbb{X}) = \infty$. We restrict attention to the case in which $\mathbb{X}$ has a countable partition, $\mathbb{X} = \bigcup_{\ell=1}^{\infty} \mathbb{X}_\ell$ with $\Lambda(\mathbb{X}_\ell) < \infty$ and $\sum_{\ell=1}^{\infty}\Lambda(\mathbb{X}_\ell) = \infty$. (Such a measure is said to be σ-finite) This would be the case, for example, if $\Lambda(A) = \int_A \lambda\, dx$, for $A \subset \mathbb{R}^d$ and λ a constant.

The construction is straightforward. For $\ell = 1, 2, \ldots$, let $\{X_{\ell 1}, X_{\ell 2}, \ldots\}$ be independent copies of iid RVs where $P(X_{\ell k} \in A) = \Lambda(A \cap \mathbb{X}_\ell)/\Lambda(\mathbb{X}_\ell)$ for $k = 1, 2, \ldots$.

Furthermore, independent of the $\{X_{\ell k}\}$, let $M_1, M_2, \ldots$ be ⫫ Poisson RVs with M_ℓ having parameter $\Lambda(\mathbb{X}_\ell)$.

Put $N_\ell(A) := \sum_{k=1}^{M_\ell} I_A(X_{\ell k})$ and

$$N(A) := \sum_{\ell=1}^{\infty} N_\ell(A) = \sum_{\ell=1}^{\infty} \sum_{k=1}^{M_\ell} I_A(X_{\ell k}).$$

By construction, the N_ℓ are $\perp\!\!\!\perp$ Poisson processes, and so $N(A)$ is a Poisson RV with parameter

$$\sum_{\ell=1}^{\infty} \Lambda(A \cap \mathbb{X}_\ell) = \Lambda\left(A \cap \left[\bigcup_{\ell=1}^{\infty} \mathbb{X}_\ell\right]\right)$$
$$= \Lambda(A \cap \mathbb{X})$$
$$= \Lambda(A).$$

If $A_1, \ldots, A_J$ are disjoint, we must show that $N(A_1), \ldots, N(A_J)$ are $\perp\!\!\!\perp$. Now, $N(A_j) = \sum_{\ell=1}^{\infty} N_\ell(A_j)$. Consider two terms from different sums, say $N_\ell(A_j)$ and $N_{\ell'}(A_{j'})$ for $j \neq j'$. If $\ell = \ell'$, then $N_\ell(A_j) \perp\!\!\!\perp N_\ell(A_{j'})$ because N_ℓ is a Poisson process & A_j and $A_{j'}$ are disjoint. Since N_ℓ & $N_{\ell'}$ are $\perp\!\!\!\perp$ for $\ell \neq \ell'$, we have $N_\ell(A_j)$ and $N_{\ell'}(A_{j'})$ $\perp\!\!\!\perp$ in this case as well. It then follows that the sums $N(A_1), \ldots, N(A_J)$ are $\perp\!\!\!\perp$.