Construction of Poisson Processes

Let Λ have no point masses and satisfy $0 < \Lambda(\mathbb{X}) < \infty$. Let $X_1, X_2, \ldots$ be i.i.d. with $\mathcal{P}(X_k \in A) = \Lambda(A)/\Lambda(\mathbb{X})$. In other words, in case Λ has a density $\lambda(x)$, X_k has pdf $\lambda(x)/\Lambda(\mathbb{X})$. The reason for requiring that Λ have no point masses is to guarantee that $X_i \neq X_j$ with probability one when $i \neq j$.

Next, independent of $\{X_k\}_{k=1}^{\infty}$, let M be independent and have a Poisson$(\Lambda(\mathbb{X}))$ pmf.

Define

$$N(A) := \sum_{k=1}^{M} I_A(X_k).$$

If $M=0$, which can happen with prob. $e^{-\Lambda(\mathbb{X})} > 0$, we put $N(A) := 0$.

We claim N is a Poisson random measure. For $g(x) \geq 0$, $\int g\,dN = \sum_{k=1}^{m} g(X_k)$, and

$$\begin{aligned}
\psi_N(g) &= E[e^{-\int g\,dN}] \\
&= \sum_{m=0}^{\infty} E[e^{-\int g\,dN} \mid M=m]\,\mathcal{P}(M=m), && \text{law of total prob.,} \\
&= \sum_{m=0}^{\infty} E[e^{-\sum_{k=1}^{m} g(X_k)}]\,\mathcal{P}(M=m), && \text{subst.,} \\
&= \sum_{m=0}^{\infty} \left(\prod_{k=1}^{m} \left[\frac{\int e^{-g(x)}\,\Lambda(dx)}{\Lambda(\mathbb{X})} \right] \right) \mathcal{P}(M=m), && \perp\!\!\!\perp,
\end{aligned}$$

$$= \sum_{m=0}^{\infty} \left[\frac{\int e^{-g(x)} \Lambda(dx)}{\Lambda(\mathbb{X})} \right]^m \cdot \frac{\Lambda(\mathbb{X})^m e^{-\Lambda(\mathbb{X})}}{m!}$$

$$= e^{-\Lambda(\mathbb{X})} \exp\left[\int e^{-g(x)} \Lambda(dx) \right]$$

$$= \exp\left[\int \{ e^{-g(x)} - 1 \} \Lambda(dx) \right].$$

By the Theorem on p. L-5, N is a Poisson random measure. ∎

We next construct a Poisson process when Λ is a σ-finite measure. i.e., when there is a partition of $\mathbb{X}$, say $\mathbb{X} = \bigcup_{\ell=1}^{\infty} \mathbb{X}_\ell$, with the property that

$\Lambda(\mathbb{X} \cap \mathbb{X}_\ell) < \infty$ for each ℓ. Then

$$\Lambda_\ell(B) := \Lambda(B \cap \mathbb{X}_\ell)$$

defines a sequence of finite measures on $\mathbb{X}$, and

$$\Lambda(B) = \Lambda(B \cap \mathbb{X}) = \Lambda(B \cap \bigcup_\ell \mathbb{X}_\ell)$$

$$= \sum_\ell \Lambda(B \cap \mathbb{X}_\ell)$$

$$= \sum_\ell \Lambda_\ell(B).$$

Let $N_1, N_2, \ldots$ be a sequence of $\perp\!\!\!\perp$ Poisson random measures on $\mathbb{X}$ with N_ℓ having Λ_ℓ as its mean measure. Define

$$N(A) := \sum_{\ell=1}^{\infty} N_\ell(A).$$

We must show N is a Poisson random measure. For $g \geq 0$, $\int g\,dN = \sum_{\ell=1}^{\infty} \int g\,dN_\ell$. Then

$$\Psi_N(g) = E[e^{-\int g\,dN}] = E\left[\prod_{\ell=1}^{\infty} e^{-\int g\,dN_\ell}\right]$$

$$= \prod_{\ell=1}^{\infty} E[e^{-\int g\,dN_\ell}]$$

$$= \prod_{\ell=1}^{\infty} \exp\left[\int \{e^{-g(x)} - 1\} \Lambda_\ell(dx)\right]$$

$$= \exp\left[\sum_{\ell=1}^{\infty} \int \{e^{-g(x)} - 1\} \Lambda_\ell(dx)\right]$$

$$= \exp\left[\sum_{\ell=1}^{\infty} \int_{\mathbb{X}_\ell} \{e^{-g(x)} - 1\} \Lambda(dx)\right]$$

$$= \exp\left[\int_{\mathbb{X}} \{e^{-g(x)} - 1\} \Lambda(dx)\right],$$

and so N is a Poisson random measure. $\blacksquare$