Marked Poisson Processes

Let N be a Poisson process whose points take values in a space $\mathbb{X}$ and having mean measure Λ. Let $\mathbb{Y}$ be another space, and let $P(dy|x)$ be a conditional probability measure on $\mathbb{Y}$. Let $X_1, X_2, \ldots$ be the points of N. Given these values, say $X_1 = x_1, \ldots$, let $Y_1, Y_2, \ldots$ be chosen conditionally independently with

$$\mathcal{P}(Y_k \in B \mid X_1 = x_1, X_2 = x_2, \ldots) = P(B \mid x_k).$$

Define a new counting measure on $\mathbb{X} \times \mathbb{Y}$ by

$$N^*(\ C\ ) := \sum_{i=1}^{\infty} I_C(X_i, Y_i).$$

<u>Theorem</u> N^* is a Poisson process on $\mathbb{X} \times \mathbb{Y}$ with mean measure

$$\Lambda^*(C) = \int_{\mathbb{X}} \left[\int_{\mathbb{Y}} I_C(x,y)\, P(dy|x) \right] \Lambda(dx).$$

"<u>Proof.</u>" Given any function $g(x,y)$, put

$$Z := \iint g(x,y)\, N^*(dx \times dy) = \sum_{i=1}^{\infty} g(X_i, Y_i)$$

We compute

$$E[e^Z] = E[E[e^Z \mid X_1, X_2, \ldots]].$$

Now,

$$E[e^Z \mid X_1 = x_1, \ldots] = E\left[e^{\sum_{i=1}^{\infty} g(x_i, Y_i)} \mid X_1 = x_1, \ldots \right]$$
$$= \prod_{i=1}^{\infty} E\left[e^{g(x_i, Y_i)} \mid X_i = x_i \right]$$

Put $h(x) := \ln E[e^{g(x,Y_i)} \mid X_i = x]$ so that

$$E[e^{Z} \mid X_1 = x_1, \ldots] = \prod_{i=1}^{\infty} e^{h(x_i)} = \exp\left[\sum_{i=1}^{\infty} h(x_i)\right].$$

Thus,

$$\begin{aligned} E[e^{Z}] &= E\left[\exp\left(\sum_{i=1}^{\infty} h(X_i)\right)\right] \\ &= E\left[e^{\int h(x)\, dN(x)}\right] \\ &= \exp\left[\int \{e^{h(x)} - 1\}\, d\Lambda(x)\right], \end{aligned}$$

where we have used the result on p. L-5. Now, since h is a log, $e^{h(x)} = E[e^{g(x,Y_i)} \mid X_i = x]$
$$= \int e^{g(x,y)}\, P(dy \mid x),$$

and so

$$\begin{aligned} E[e^{Z}] &= \exp\left[\int \left\{\int e^{g(x,y)}\, P(dy \mid x) - 1\right\} d\Lambda(x)\right] \\ &= \exp\left[\int \int \{e^{g(x,y)} - 1\} P(dy \mid x)\, d\Lambda(x)\right]. \\ &= \exp\left[\iint \{e^{g(x,y)} - 1\}\, \Lambda^{*}(dx \times dy)\right] \end{aligned}$$

Since $E[e^{Z}]$ is the pgfl, we have by the theorem on p. L5 that N^{*} is a Poisson counting measure on $\mathbb{X} \times \mathbb{Y}$. $\boxed{}$

NOTE: $N^{*}(A \times \mathbb{Y}) = \sum_{i=1}^{\infty} I_{A \times \mathbb{Y}}(X_i, Y_i) = \sum_{i=1}^{\infty} I_A(X_i) \overbrace{I_{\mathbb{Y}}(Y_i)}^{=1} = \sum_{i=1}^{\infty} I_A(X_i)$
$= N(A).$