Cluster Processes

A *cluster process* N on a space $\mathbb{Y}$ is constructed from a *center process* N_c on a space $\mathbb{X}$ and a *family of component processes* $\{N_r(\cdot|x), x\in\mathbb{X}\}$.

We put

$$N(B) := \int N_r(B|x)\,N_c(dx), \quad B\subset \mathbb{Y}.$$

If $\Lambda_c(A) := E[N_c(A)]$ for $A\subset\mathbb{X}$, and $\Lambda_r(B|x) := E[N_r(B|x)]$ for $x\in\mathbb{X}$, $B\subset\mathbb{Y}$, then $E[N(B)] = \int \Lambda_r(B|x)\Lambda_c(dx)$

If we denote the points of N_c by $\{X_i\}_{i=1}^{\infty}$, then

$$N(B) = \sum_i N_r(B|X_i).$$

Def. For $g(y)\geq 0$, put

$$\Psi_r(g|x) := E\left[e^{-\int g(y)\,N_r(dy|x)}\right].$$

We always assume this – In this case, N is called an independent cluster process

Prop. If the component processes $\{N_r(\cdot|x)\}$ are mutually independent and independent of the center process N_c, then

$$E\left[e^{-\int g(y)N(dy)}\,\middle|\,N_c(\cdot)\right] = \exp\left[\int \ln\Psi_r(g|x)\,N_c(dx)\right].$$

Proof. Suppose the points of N_c are $X_i = x_i$. Then

$$\int g(y)N(dy) = \sum_i \int g(y)N_r(dy|x_i), \text{ and}$$

$$E\left[e^{-\int g(y)N(dy)}\,\middle|\,X_i=x_i \text{ all } i\right] = E\left[\prod_i e^{-\int g(y)N_r(dy|x_i)}\right]$$

$$= \prod_i \Psi_r(g|x_i) = \exp\left(\sum_i \ln\Psi_r(g|x_i)\right)$$

Then

$$E[e^{-\int g(y)N(dy)} | N_c(\cdot)] = \exp\left[\sum_i \ln \Psi_r(g|X_i)\right]$$

$$= \exp\left[\int \ln \Psi_r(g|x)\, N_c(dx)\right].$$

■

<u>Corollary</u> If the center process of an independent cluster process is Poisson, with mean measure Λ_c, then

$$\Psi(g) := E[e^{-\int g(y)N(dy)}] = \exp\left[\int \{\Psi_r(g|x) - 1\}\, \Lambda_c(dx)\right].$$

<u>Proof.</u> Write

$$\Psi(g) = E[e^{-\int g(y)N(dy)}]$$

$$= E[\,E[e^{-g(y)N(dy)} | N_c(\cdot)]\,]$$

$$= E[e^{sY}]\Big|_{s=1}$$

where

$$Y := \int h(x)\, N_c(dx) \quad \text{and} \quad h(x) := \ln \Psi_r(g|x).$$

Thus,

$$\Psi(g) = \exp\left[\int \{e^{sh(x)} - 1\}\, \Lambda_c(dx)\right]\Big|_{s=1}$$

$$= \exp\left[\int \{\Psi_r(g|x) - 1\}\, \Lambda_c(dx)\right].$$

■

If $N_r(\cdot|x)$ is Poisson with mean measure $\Lambda_r(\cdot|x)$, then

$$\Psi_r(g|x) = \exp\left[\int \{e^{-g(y)} - 1\}\, \Lambda_r(dy|x)\right].$$

Now suppose we put $Y = \int h(y) N(dy)$. Then $M_Y(s) := E[e^{sY}] = \Psi(g)$ if $g(y) = -sh(y)$. Thus, $M_Y(s) = \Psi(sh)$. When N_c is Poisson,

$$M_Y(s) = \Psi(sh) = \exp\left[\int \{\Psi_r(-sh|x) - 1\} \Lambda_c(dx)\right]$$

If $N_r(\cdot|x)$ is Poisson as well,

$$\Psi_r(-sh|x) = \exp\left[\int \{e^{+sh(y)} - 1\} \Lambda_r(dy|x)\right].$$

Then

$$\frac{d}{ds} \Psi_r(-sh|x) = \Psi_r(-sh|x) \cdot \int h(y) e^{sh(y)} \Lambda_r(dy|x)$$

$$\frac{d}{ds} \Psi_r(-sh|x)\Big|_{s=0} = \int h(y) \Lambda_r(dy|x).$$

Furthermore,

$$M_Y'(s) = M_Y(s) \cdot \int \frac{d}{ds} \Psi_r(-sh|x)\, \Lambda_c(dx)$$

$$M_Y'(0) = \int \left[\int h(y) \Lambda_r(dy|x)\right] \Lambda_c(dx)$$

Thus,

$$E\left[\int h(y) N(dy)\right] = \int \left[\int h(y) \Lambda_r(dy|x)\right] \Lambda_c(dx).$$

Let us continue with second derivatives. Then

$$M_Y''(s) = M_Y(s) \left(\int \frac{d}{ds} \Psi_r(-sh|x)\, \Lambda_c(dx)\right)^2 + M_Y(s) \cdot \int \frac{d^2}{ds^2} \Psi_r(-sh|x)\, \Lambda_c(dx).$$

We also need

$$\frac{d^2}{ds^2}\Psi_r(-sh|x) = \Psi_r(-sh|x)\cdot\left(\int h(y)e^{sh(y)}\Lambda_r(dy|x)\right)^2$$
$$+\Psi_r(-sh|x)\int h(y)^2 e^{sh(y)}\Lambda_r(dy|x).$$

Then

$$M_Y''(0) = \left(\int\left[\int h(y)\Lambda_r(dy|x)\right]\Lambda_c(dx)\right)^2$$
$$+\int\left[\left(\int h(y)\Lambda_r(dy|x)\right)^2 + \int h(y)^2\Lambda_r(dy|x)\right]\Lambda_c(dx).$$

Thus,

$$E\left[\left(\int h(y)N(dy)\right)^2\right] = \left(E\left[\int h(y)N(dy)\right]\right)^2$$
$$+\int\left[\int h(y)\Lambda_r(dy|x)\right]^2\Lambda_c(dx)$$
$$+\int\left[\int h(y)^2\Lambda_r(dy|x)\right]\Lambda_c(dx),$$

and so

$$\operatorname{var}\left(\int h(y)N(dy)\right) = \int\left[\int h(y)\Lambda_r(dy|x)\right]^2\Lambda_c(dx)$$
$$+\int\left[\int h(y)^2\Lambda_r(dy|x)\right]\Lambda_c(dx).$$

We also need $E\left[\left\{\int p(y)N(dy)\right\}\left\{\int q(y)N(dy)\right\}\right]$.

Let $g(y) = -s_1 p(y) - s_2 q(y)$ so that if

$$P := \int p(y)N(dy) \quad \text{and} \quad Q := \int q(y)N(dy),$$

then

$$M(s_1,s_2) := E[e^{s_1P+s_2Q}]$$
$$= E[e^{-\int g(y)N(dy)}]$$

and

$$E[PQ] = \frac{\partial^2}{\partial s_2 \partial s_1} M(s_1,s_2)\Big|_{s_1=s_2=0}.$$

Now,

$$M(s_1,s_2) = \exp\left[\int \{\Psi_r(g|x)-1\}\Lambda_c(dx)\right],$$

where now

$$\Psi_r(g|x) = \exp\left[\int\{e^{s_1p(y)+s_2q(y)}-1\}\Lambda_r(dy|x)\right].$$

So,

$$\frac{\partial}{\partial s_1}\Psi_r(g|x) = \Psi_r(g|x)\cdot\int p(y)e^{s_1p(y)+s_2q(y)}\Lambda_r(dy|x),$$

and

$$\frac{\partial}{\partial s_1}M(s_1,s_2) = M(s_1,s_2)\int\frac{\partial}{\partial s_1}\Psi_r(g|x)\,\Lambda_c(dx).$$

Next,

$$\frac{\partial^2}{\partial s_2\partial s_1}\Psi_r(g|x) = \Psi_r(g|x)\int p(y)e^{s_1p(y)+s_2q(y)}\Lambda_r(dy|x)$$
$$\cdot\int q(y)e^{s_1p(y)+s_2q(y)}\Lambda_r(dy|x)$$
$$+\Psi_r(g|x)\cdot\int p(y)q(y)e^{s_1p(y)+s_2q(y)}\Lambda_r(dy|x)$$

and so

$$\frac{\partial^2}{\partial s_2 \partial s_1} M(s_1,s_2) = M(s_1,s_2) \int \frac{\partial}{\partial s_1} \psi_r(g|x)\Lambda_c(dx) \int \frac{\partial}{\partial s_2} \psi_r(g|x)\Lambda_c(dx)$$
$$+ M(s_1,s_2) \int \frac{\partial^2}{\partial s_2 \partial s_1} \psi_r(g|x)\Lambda_c(dx).$$

Then

$$E[PQ] = \int \left[\int p(y)\Lambda_r(dy|x) \right] \Lambda_c(dx) \cdot \int \left[\int q(y)\Lambda_r(dy|x) \right] \Lambda_c(dx)$$
$$+ \int \left[\int p(y)\Lambda_r(dy|x) \cdot \int q(y)\Lambda_r(dy|x) \right] \Lambda_c(dx)$$
$$+ \int \left[\int p(y)q(y)\Lambda_r(dy|x) \right] \Lambda_c(dx).$$

<u>Notation</u> Define the linear functional $\bar{\Lambda}_c$ on functions $v(x)$ by

$$\bar{\Lambda}_c v := \int v(x)\Lambda_c(dx).$$

Define the linear transformation $\bar{\Lambda}_r$ that takes functions $p(y)$ and returns functions of x by

$$(\bar{\Lambda}_r p)(x) := \int p(y)\Lambda_r(dy|x).$$

Then

$$E[P] := E\left[\int p(y)N(dy)\right] = \bar{\Lambda}_c(\bar{\Lambda}_r p).$$

$$var(P) = \bar{\Lambda}_c((\bar{\Lambda}_r p)^2) + \bar{\Lambda}_c(\bar{\Lambda}_r(p^2)).$$

$$E[PQ] = \bar{\Lambda}_c(\bar{\Lambda}_r p)\cdot\bar{\Lambda}_c(\bar{\Lambda}_r q) + \bar{\Lambda}_c((\bar{\Lambda}_r p)\cdot(\bar{\Lambda}_r q)) + \bar{\Lambda}_c(\bar{\Lambda}_r(p\cdot q)).$$

$$\mathrm{cov}(P,Q) = \bar{\Lambda}_c((\bar{\Lambda}_r p)\cdot(\bar{\Lambda}_r q)) + \bar{\Lambda}_c(\bar{\Lambda}_r(p\cdot q)).$$

$$\psi(g) = \exp\left[\bar{\Lambda}_c\left(\exp\{\bar{\Lambda}_r(e^{-g}-1)\}-1\right)\right].$$

For the center process N_c, if

$$V := \int v(x)\,N_c(dx) \quad\text{and}\quad W := \int w(x)\,N_c(dx),$$

then

$$E[V] = \int v(x)\,\Lambda_c(dx) = \bar{\Lambda}_c v,$$

$$E[V^2] = \int v(x)^2\,\Lambda_c(dx) + \left(\int v(x)\,\Lambda_c(dx)\right)^2 = \bar{\Lambda}_c(v^2) + (\bar{\Lambda}_c v)^2,$$

$$E[VW] = \int v(x)w(x)\,\Lambda_c(dx) + \int v(x)\,\Lambda_c(dx)\int w(x)\,\Lambda_c(dx) = \bar{\Lambda}_c(v\cdot w) + (\bar{\Lambda}_c v)\cdot(\bar{\Lambda}_c w),$$

$$\mathrm{cov}(V,W) = \int v(x)w(x)\,\Lambda_c(dx) = \bar{\Lambda}_c(v\cdot w),$$

$$E[e^V] = \exp\left[\bar{\Lambda}_c(e^v - 1)\right].$$

The last quantity we compute is

$$E[VP] = E[E[VP \mid N_c(\cdot)]]$$
$$= E[V E[P \mid N_c(\cdot)]]$$

Next, if $X_1 = x_1, X_2 = x_2, \ldots$, then

$$E[P \mid X_1 = x_1, \ldots] = E\left[\sum_i \int p(y) N_r(dy \mid x_i)\right]$$
$$= \sum_i E\left[\int p(y) N_r(dy \mid x_i)\right]$$
$$= \sum_i \int p(y) \Lambda_r(dy \mid x_i)$$
$$= \sum_i (\bar{\Lambda}_r p)(x_i).$$

So,

$$E[VP] = E\left[V \cdot \sum_i (\bar{\Lambda}_r P)(X_i)\right]$$
$$= E\left[V \cdot \int (\bar{\Lambda}_r P)(x) N_c(dx)\right]$$
$$= E\left[\int v(x) N_c(dx) \cdot \int (\bar{\Lambda}_r P)(x) N_c(dx)\right]$$
$$= \bar{\Lambda}_c(v \cdot \bar{\Lambda}_r p) + \bar{\Lambda}_c v \cdot \bar{\Lambda}_c(\bar{\Lambda}_r p).$$

One can also show that

$$E[e^{s(V+P)}] = \exp\left[\bar{\Lambda}_c\left(\exp(sv + \bar{\Lambda}_r\{e^{sp} - 1\}) - 1\right)\right].$$