Augmented Cluster Processes

So far, we have been dealing with two different spaces $\mathbb{X}$ and $\mathbb{Y}$. However, if $\mathbb{X} = \mathbb{Y}$, it makes sense to consider the <u>augmented cluster process</u>,

$$N_a(B) := N_c(B) + N(B).$$

In other words, the points of the cluster center process N_c are "added" to the points of the cluster process N. Of course, any integral (shot noise RV)

$$H := \int h \, dN_a = \int h \, dN_c + \int h \, dN$$

has the form $V + P$, where $V = \int h \, dN_c$ and $P = \int h \, dN$. When N_c and $N_r(\cdot | x)$ are independent and Poisson it is easy to show that

$$E[H] = \bar{\Lambda}_c h + \bar{\Lambda}_c(\Lambda_r h)$$

$$\begin{aligned} \mathrm{var}(H) = {} & \bar{\Lambda}_c(h^2) + 2\bar{\Lambda}_c(h \cdot \bar{\Lambda}_r h) \\ & + \bar{\Lambda}_c(\bar{\Lambda}_r(h^2)) + \bar{\Lambda}_1((\bar{\Lambda}_r h)^2). \end{aligned}$$

$$E[e^{sH}] = \exp\left[\bar{\Lambda}_c\left(\exp(sh + \bar{\Lambda}_r\{e^{sh} - 1\}) - 1\right)\right].$$

The Saleh-Valenzuela Model - The NLOS Case

We take for N_c the two-dimensional Poisson process with intensity function

$$\lambda_c(\tau, \gamma) = C \cdot f_{\tau,\tau}(\gamma), \quad \tau \geq 0,$$

where $f_{\tau,\tau}(\gamma)$ is defined later. In other words, N_c is a marked temporal Poisson process in which the temporal process is a homogeneous Poisson process with constant intensity C. The mark (gain) G of a path arriving at τ has density $f_{\tau,\tau}(g)$ with second moment $\Omega_0 e^{-\tau/\tau_0}$, where Ω_0 and τ_0 are model parameters.

Next, recall that $N(B) = \int N_r(B|x) N_c(dx)$. For the SV model, for $x = (\tau, \gamma)$, we take $N_r(\cdot|\tau, \gamma)$ to be a two-dimensional Poisson process with intensity

$$\lambda_r(s, g|\tau, \gamma) = R \cdot f_{\tau,s}(g), \quad s \geq \tau.$$

In other words, $N_r(\cdot|\tau, \gamma)$ is a marked temporal Poisson process in which the temporal process is a homogeneous Poisson process with constant intensity R for $s \geq \tau$; i.e., the process starts at τ rather than at zero. The mark G of a path arriving at s has density $f_{\tau,s}(g)$ with second moment $\Omega_0 e^{-\tau/\tau_0} e^{-(s-\tau)/s_0}$, where s_0 is another parameter; τ_0 and s_0 are <u>power-delay time constants.</u> Note that $\lambda_r(s,g|\tau,\gamma)$ does <u>not</u> depend on γ, hence, neither does $N_r(\cdot|\tau,\gamma)$.

The augmented cluster process is $N_a(B) = N_c(B) + N(B)$, where $N(B) = \iint N_r(B|\tau,\gamma)\, N_c(d\tau \times d\gamma)$ and $B \subset [0,\infty) \times \mathbb{R}$.

The idea here is that N_c is the "cluster-start process," i.e., if $S_{10}, S_{20}, \ldots$ denote the arrival times of N_c, then S_{i0} is the arrival time of the initial path of the ith cluster. Let $G_{10}, G_{20}, \ldots$ denote the corresponding gains. The pairs (S_{i0}, G_{i0}) are the points of two-dimensional process N_c. Thus,

$N_c([t_1,t_2],[g_1,g_2])$ = # paths of N_c that arrive betw. t_1 & t_2 AND whose gains lie betw. g_1 & g_2

Next, given that $S_{i0} = \tau_i$ and $G_{i0} = \gamma_i$, $N_r(\cdot|\tau_i,\gamma_i)$ is the process of the remaining noninitial paths of the ith cluster. Let $S_{i1}, S_{i2}, \ldots$ denote the arrival times of these paths. Let $G_{i1}, G_{i2}, \ldots$ denote the corresponding gains.

Thus, the points of the ith cluster are the pairs $\{(S_{i0}, G_{i0}), (S_{i1}, G_{i1}), \dots\}$, where the first pair comes from N_c and the remaining pairs come from $N_r(\cdot \mid \tau_i, \gamma_i)$ given that $S_{i0} = \tau_i$ and $G_{i0} = \gamma_i$. We can now write

$$N_a(B) = N_c(B) + N(B)$$

$$= \sum_{i=1}^{\infty} I_B(S_{i0}, G_{i0}) + \sum_{i=1}^{\infty} \underbrace{\left[\sum_{j=1}^{\infty} I_B(S_{ij}, G_{ij}) \right]}_{= N_r(B \mid S_{i0}, G_{i0})}$$

More generally, given a function $\varphi(t,g)$, put

$$\Phi_a := \iint \varphi(t,g)\, N_a(dt \times dg)$$

$$= \iint \varphi(t,g)\, N_c(dt \times dg) + \iint \varphi(t,g)\, N(dt \times dg)$$

$$= \sum_{i=1}^{\infty} \varphi(S_{i0}, G_{i0}) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi(S_{ij}, G_{ij})$$

$$= \quad V \quad + \quad P.$$

__Examples__

1) $\varphi(t,g) = I_{[a,b]}(t) \Rightarrow \Phi_a = \sum_{i=1}^{\infty} I_{[a,b]}(S_{i0}) + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} I_{[a,b]}(S_{ij})$

$= \#$ paths arriving in $[a,b]$

$E[\Phi_a] =$ expected $\#$ paths in $[a,b]$

2) $\phi(t,g) = I_{[0,T]}(t)\, g \Rightarrow$

$$\Phi_a = \sum_{i=1}^{\infty} I_{[0,T]}(S_{i0})\, G_{i0} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} I_{[0,T]}(S_{ij})\, G_{ij}$$

= sum of gains of paths arriving in $[0,T]$

Compute $E[\Phi_a]$

3) $\phi(t,g) = I_{[0,T]}(t)\, g^2 \Rightarrow$

Φ_a = sum of squares of gains in $[0,T]$

$E[\Phi_a]$ = power-delay cdf

$\frac{d}{dT} E[\Phi_a]$ = power-delay profile.

Now,

$$E[\Phi_a] = \bar{\Lambda}_c \phi + \bar{\Lambda}_c(\bar{\Lambda}_r \phi)$$

$$= \int\left[\int \phi(\tau,\gamma)\, \lambda_c(\tau,\gamma)\, d\gamma\right] d\tau$$

$$+ \bar{\Lambda}_c(\bar{\Lambda}_r \phi),$$

where

$$(\bar{\Lambda}_r \phi)(\tau,\gamma) = \int\left[\int \phi(s,g)\, \lambda_r(s,g|\tau,\gamma)\, dg\right] ds$$

Let's compute

$$\bar{\Lambda}_c \phi = \int\left[\int I_{[0,T]}(\tau)\, \gamma^2 \cdot C\, f_{\Gamma,\tau}(\gamma)\, d\gamma\right] d\tau$$

$$= C\int_0^T \left[\gamma^2 f_{\Gamma,\tau}(\gamma)\, d\gamma\right] d\tau$$

$$= C\int_0^T \Omega_0\, e^{-\tau/\tau_0}\, d\tau = C\,\Omega_0 \tau_0 (1 - e^{-T/\tau_0}).$$

Next,

$$(\bar\Lambda_r\varphi)(\tau,\gamma) = R\int_\tau^\infty \left[\int I_{[0,T]}(s) g^2 f_{\tau,s}(g)\, dg\right] ds,$$

which is zero if $\tau > T$. For $\tau \le T$,

$$(\bar\Lambda_r\varphi)(\tau,\gamma) = R\int_\tau^T \left[g^2 f_{\tau,s}(g)\, dg\right] ds$$

$$= R\Omega_0 e^{-\tau/\tau_0} \int_\tau^T e^{-(s-\tau)/s_0}\, ds$$

$$= R\Omega_0\, e^{-\tau(1/\tau_0 - 1/s_0)} \int_\tau^T e^{-s/s_0}\, ds$$

$$= R\Omega_0 s_0\, e^{-\tau(1/\tau_0 - 1/s_0)} \left[e^{-\tau/s_0} - e^{-T/s_0}\right]$$

$$= R\Omega_0 s_0 \left\{ e^{-\tau/\tau_0} - e^{-\tau(1/\tau_0 - 1/s_0)} e^{-T/s_0} \right\},$$

which does NOT depend on γ. So we write $(\bar\Lambda_r\varphi)(\tau)$ instead. Thus,

$$\bar\Lambda_c(\bar\Lambda_r\varphi) = \int \left[\int (\bar\Lambda_r\varphi)(\tau)\cdot C f_{\tau,\tau}(\gamma)\, d\gamma\right] d\tau$$

$$= C\int_0^\infty (\bar\Lambda_r\varphi)(\tau)\, d\tau$$

$$= CR\Omega_0 s_0 \int_0^T e^{-\tau/\tau_0} - e^{-\tau(1/\tau_0 - 1/s_0)} e^{-T/s_0}\, d\tau$$

$$= CR\Omega_0 s_0 \left\{ \tau_0\left[1 - e^{-T/\tau_0}\right] - \frac{s_0\tau_0}{s_0-\tau_0}\left(1 - e^{-T(1/\tau_0 - 1/s_0)}\right) \cdot e^{-T/s_0} \right\}$$

$$= CR\Omega_0 s_0 \left\{ \tau_0\left[1 - e^{-T/\tau_0}\right] - \frac{s_0\tau_0}{s_0-\tau_0}\left[e^{-T/s_0} - e^{-T/\tau_0}\right] \right\}$$

$$= \Omega_0 CR s_0 \left\{ \mathcal{S}_{\tau_0}(T) - \frac{s_0\tau_0}{s_0-\tau_0}\left[e^{-T/s_0} - 1 + 1 - e^{-T/\tau_0}\right] \right\}$$

where $\mathcal{S}_\mu(t) := \mu\left[1 - e^{-t/\mu}\right]$.

Then

$$\bar{\Lambda}_c(\bar{\Lambda}_n \varphi) = \Omega_0 C R s_0 \left\{ \varsigma_{\tau_0}(T) - \frac{s_0 \tau_0}{s_0-\tau_0}\left[-\frac{1}{s_0}\varsigma_{s_0}(T) + \frac{1}{\tau_0}\varsigma_{\tau_0}(T)\right]\right\}$$

$$= \Omega_0 C R s_0 \left\{ \varsigma_{\tau_0}(T) + \frac{\tau_0}{s_0-\tau_0}\varsigma_{s_0}(T) - \frac{s_0}{s_0-\tau_0}\varsigma_{\tau_0}(T)\right\}$$

$$= \Omega_0 C R s_0 \left\{ \frac{-\tau_0}{s_0-\tau_0}\varsigma_{\tau_0}(T) + \frac{\tau_0}{s_0-\tau_0}\varsigma_{s_0}(T)\right]$$

$$= \Omega_0 C R \frac{s_0\tau_0}{s_0-\tau_0}\left\{ \varsigma_{s_0}(T) - \varsigma_{\tau_0}(T)\right\}.$$

Finally, the pd-cdf is

$$P(T) = \Omega_0 C \left\{ \varsigma_{\tau_0}(T) + \frac{R s_0 \tau_0}{s_0-\tau_0}\left[\varsigma_{s_0}(T) - \varsigma_{\tau_0}(T)\right]\right\}$$

and the pdp is

$$p(t) = \Omega_0 C \left\{ e^{-t/\tau_0} + \frac{R s_0\tau_0}{s_0-\tau_0}\left[e^{-t/s_0} - e^{-t/\tau_0}\right]\right\}$$

$$= \Omega_0 C \left\{ \left(1 - \frac{R s_0 \tau_0}{s_0-\tau_0}\right) e^{-t/\tau_0} + \frac{R s_0\tau_0}{s_0-\tau_0} e^{-t/s_0}\right\}.$$

To convert the pdp to a probability density, we must divide it by $\lim_{T\to\infty} P(T) = \Omega_0 C\left\{ \tau_0 + \frac{R s_0\tau_0}{s_0-\tau_0}[s_0-\tau_0]\right\}$

$$= \Omega_0 C \tau_0 (1 + R s_0).$$

Also, on account of the exponential form of $p(t)$, it is easy to find the mean excess delay and the delay spread:

$$\bar{D} = \frac{1}{1+Rs_0}\left\{ \left(1 - \frac{R s_0\tau_0}{s_0-\tau_0}\right)\cdot \tau_0 + \frac{R s_0^2}{s_0-\tau_0} s_0 \right\}$$

$$= \frac{1}{1+Rs_0}\left\{ \tau_0 - \frac{R s_0 \tau_0^2}{s_0-\tau_0} + \frac{R s_0^3}{s_0-\tau_0}\right\}$$

$$= \frac{1}{1+RS_o}\left\{\tau_o + \frac{RS_o}{S_o-\tau_o}\left(S_o^2-\tau_o^2\right)\right\}$$

$$= \frac{1}{1+RS_o}\left\{\tau_o + RS_o\left(S_o+\tau_o\right)\right\}$$

$$= \frac{1}{1+RS_o}\left\{\tau_o\left(1+RS_o\right) + RS_o^2\right\} = \tau_o + \frac{RS_o}{1+RS_o}S_o .$$

Similarly,

$$\overline{D^2} = 2\left[\tau_o^2 + \frac{RS_o}{1+RS_o}S_o\left(S_o+\tau_o\right)\right].$$

Finally, the delay spread is

$$S = \sqrt{\overline{D^2}-\left(\overline{D}\right)^2} = \left[\tau_o^2 + \frac{2RS_o^3}{1+RS_o} - \left(\frac{RS_o^2}{1+RS_o}\right)^2\right]^{1/2}.$$

For large R, $\overline{D} \simeq \tau_o + S_o$ and $S \approx \sqrt{\tau_o^2+S_o^2}$.

4) Again with $\phi(t;g) = I_{[0,T]}(t) g^2$,
let's compute the MGF $M_a(\theta) = E[e^{\theta \Phi_a}]$.
First,

$$(\bar{\Lambda}_r\{e^{\theta\phi}-1\})(\tau,\gamma) = R\int_\tau^\infty \left[\int \{e^{\theta I_{[0,T]}(s) g^2} - 1\} f_{\tau,s}(g)\,dg\right] ds.$$

For $\tau > T$, this is zero. For $0 \le \tau \le T$, it is

$$R\int_\tau^T \int \{e^{\theta g^2} - 1\} f_{\tau,s}(g)\,dg\,ds = R\int_\tau^T [M_{\tau,s}(\theta) - 1]\,ds$$

does not depend on γ, and $M_{\tau,s}(\theta) := E_{\tau,s}[e^{\theta G^2}] = \int e^{\theta g^2} f_{\tau,s}(g)\,dg$.
Next,

$$\theta I_{[0,T]}(\tau)\gamma^2 + (\bar{\Lambda}_r\{e^{\theta\phi}-1\})(\tau)$$

is zero for $\tau > T$. Thus,

$$M_a(\theta) = \exp\left[c\int_0^T \int \left\{e^{\theta\gamma^2 + (\bar{\Lambda}_r\{e^{\theta\phi}-1\})(\tau)} - 1\right\} f_{\tau,\tau}(g)\,dg\,d\tau\right]$$

$$= \exp\left[c\int_0^T \left\{M_{\tau,\tau}(\theta)\, e^{R\int_\tau^T M_{\tau,s}(\theta) - 1\,ds} - 1\right\} d\tau\right].$$

We could now use $M_a(\theta)$ in Craig's formula to compute $P_T(\eta)$. for a channel modeled as an augmented cluster process in which the cluster centers and the components are Poisson processes, as in the IEEE 802.15.3a model.