ECE 901 Notes & HW # 9
Multipath-Cluster Channel Models

I. Introduction

In multipath channels, paths often arrive in clusters. Let T_{i0} denote the arrival time of the initial path of the ith cluster. We call the T_{i0} **cluster-start times**. In line-of-sight (LOS) channels, we start indexing at $i=0$ with $T_{00} \equiv 0$. In non-LOS (NLOS) channels, we start indexing at $i=1$. In either case, we denote the occurrence times of the noninitial paths of the ith cluster by $T_{i1}, T_{i2}, \ldots$. Every path, whether an initial path of a cluster or a noninitial path of a cluster, has an associated gain $G_{ij} \geq 0$ and a phase $\Theta_{ij} \in [0, 2\pi)$. Additional quantities may also be associated with each occurrence time; e.g., an angle of arrival, an angle of departure, polarization information, etc.

II. A Model for Multipath Clustering

As mentioned earlier, a path arriving at time T_{ij} has an associated gain G_{ij} and phase Θ_{ij}, and possibly additional quantities as well. All of these associated quantities are called **marks**. We collect them into a mark vector Y_{ij}. Our general model allows the mark vector of the initial path of a cluster to be longer than the mark vectors of the noninitial paths of the cluster.

A. The Joint Distributions

To specify the joint distributions of the random variables $\{(T_{ij}, Y_{ij})\}$, we proceed as follows. First, we specify the distributions of the (T_{i0}, Y_{i0}) for all i. Second, we specify the conditional distributions of the (T_{ij}, Y_{ij}) for $j \geq 1$, given the values of (T_{i0}, Y_{i0}) for all i. In the theory of cluster point processes [1], the collection of points $\{(T_{i0}, Y_{i0})\}$ is called the **cluster-center process**, while the collection of points $\{(T_{ij}, Y_{ij}),$ all i and all $j \geq 1\}$ is the **cluster process** itself.

1) The (T_{i0}, Y_{i0}): The T_{i0} are taken to be the occurrence times of a temporal point process. Given that $T_{i0} = t_i$ for all i, we take the Y_{i0} to be conditionally independent, with each Y_{i0} depending only on the corresponding value t_i, but not depending on the index i. Then the pairs (T_{i0}, Y_{i0}) constitute a **marked point process** [1], [3].

2) The (T_{ij}, Y_{ij}) for $j \geq 1$: Given that $(T_{i0}, Y_{i0}) = (t_i, y_i)$ for all i, we take the collections $\{(T_{ij}, Y_{ij})\}_{j\geq 1}$ to be conditionally independent, with the ith collection depending only on the value of (t_i, y_i), but not depending on the index i.

It remains to specify the conditional distribution of the ith collection, given that $(T_{i0}, Y_{i0}) = (t, y)$. Given that $T_{i0} = t$ and $Y_{i0} = y$, we take $\{T_{ij}\}_{j\geq 1}$ to be the occurrence times of a temporal point process starting at time t with no dependence on y or the index i. Given further that $T_{ij} = \tau_{ij}$ for all $j \geq 1$, we take the Y_{ij} for $j \geq 1$ to be conditionally independent, with each Y_{ij} depending only on the corresponding values of τ_{ij}, t, and y, but not on the indexes i and j. Then the pairs $\{(T_{ij}, Y_{ij})\}_{j\geq 1}$ form a marked point process conditional on $(T_{i0}, Y_{i0}) = (t, y)$.

III. Characterization of the Model

Because Y_{i0} and Y_{ij} for $j \geq 1$ can have different numbers of components, we consider sums of the form

$$Z = \sum_i \alpha(T_{i0}, Y_{i0}) + \sum_i \sum_{j\geq 1} \beta(T_{ij}, Y_{ij}) \tag{1}$$

Below we characterize $\mathsf{E}[e^Z]$. Of course, by replacing α and β by $s\alpha$ and $s\beta$, we immediately obtain the moment generating function of Z, which can then be differentiated to obtain the moments of Z.

A. Laplace Functionals

The **Laplace functional** of the cluster-center point process $\{(T_{i0}, Y_{i0})\}$ is defined on functions $\alpha(t, y)$ by

$$\mathbf{L}_c^* \alpha := \mathsf{E}\big[e^{\sum_i \alpha(T_{i0}, Y_{i0})}\big].$$

Similarly, the Laplace functional of the cluster process $\{(T_{ij}, Y_{ij}),$ all i and all $j \geq 1\}$ is defined on functions $\beta(t, y)$ by

$$\mathbf{L}^* \beta := \mathsf{E}\big[e^{\sum_i \sum_{j\geq 1} \beta(T_{ij}, Y_{ij})}\big].$$

For any statistic Z as in (1) above, we can use the law of total probability and our conditional independence assumptions to show that

$$\mathsf{E}[e^Z] = \mathbf{L}_c^*(\alpha + \log \widehat{\mathbf{L}}^* \beta), \tag{2}$$

where $\widehat{\mathbf{L}}^*$ is the conditional Laplace functional,

$$(\widehat{\mathbf{L}}^* \beta)(t, y) := \mathsf{E}\big[e^{\sum_{j\geq 1} \beta(T_{ij}, Y_{ij})} \big| T_{i0} = t, Y_{i0} = y\big].$$

By taking $\alpha \equiv 0$ in (2), we obtain the special case

$$\mathbf{L}^* \beta = \mathbf{L}_c^*(\log \widehat{\mathbf{L}}^* \beta).$$

Because our point processes are marked, we can express the Laplace functionals $\mathbf{L}_c^*$ and $\widehat{\mathbf{L}}^*$, which are defined on functions of pairs (t, y), in terms of Laplace functionals on functions of time only. A well-known result for marked point processes [2, p. 17, Example 1.28], which is easy to show using the law of total probability and our conditional independence assumptions, is that

$$\mathbf{L}_c^* \alpha = \mathbf{L}_c(\log K_c \alpha),$$

where, on functions $v(t)$, $\mathbf{L}_c$ is the Laplace functional of the temporal point process $\{T_{i0}\}$,

$$\mathbf{L}_c v := \mathsf{E}\big[e^{\sum_i v(T_{i0})}\big], \tag{3}$$

and K_c is the operator

$$(K_c\alpha)(t) := \mathsf{E}\big[e^{\alpha(t,Y_{i0})}\big|T_{i0}=t\big]. \tag{4}$$

It then follows that

$$\mathsf{E}[e^Z] = L_c(K_c\{\alpha+\log\widehat{\mathbf{L}}^*\beta\}), \tag{5}$$

where

$$[K_c\{\alpha+\log\widehat{\mathbf{L}}^*\beta\}](t) = \mathsf{E}\big[e^{\alpha(t,Y_{i0})}(\widehat{\mathbf{L}}^*\beta)(t,Y_{i0})\big|T_{i0}=t\big]. \tag{6}$$

Just as we wrote $\mathbf{L}_c^*$ in terms of $\mathbf{L}_c$ and K_c, it can be shown that

$$(\widehat{\mathbf{L}}^*\beta)(t,y) = \big[\widehat{\mathbf{L}}\log(K\beta)(t,y,\cdot)\big](t,y)$$

where, on functions $v(\tau)$, $\widehat{\mathbf{L}}$ is the conditional Laplace functional of the temporal point process $\{T_{ij}\}_{j\geq 1}$,

$$(\widehat{\mathbf{L}}v)(t,y) := \mathsf{E}\big[e^{\sum_{j\geq 1}v(T_{ij})}\big|T_{i0}=t, Y_{i0}=y\big], \tag{7}$$

and K is the operator

$$(K\beta)(t,y,\tau) := \mathsf{E}\big[e^{\beta(\tau,Y_{ij})}\big|T_{i0}=t, Y_{i0}=y, T_{ij}=\tau\big]. \tag{8}$$

It then follows that

$$(\widehat{\mathbf{L}}^*\beta)(t,y) = \mathsf{E}\big[e^{\sum_{j\geq 1}\log(K\beta)(t,y,T_{ij})}\big|T_{i0}=t, Y_{i0}=y\big]. \tag{9}$$

B. Discussion

The specification of multipath-cluster channels decouples into temporal and spatial parts. The temporal behavior is determined by the cluster-start process $\{T_{i0}\}$ and the conditional point processes $\{T_{ij}\}_{j\geq 1}$. These processes are characterized by the temporal Laplace functionals $\mathbf{L}_c$ and $\widehat{\mathbf{L}}$ in (3) and (7), which do *not* involve α and β. The spatial behavior is determined by the operators K_c and K, which are defined by the conditional expectations in (4) and (8), and *do* involve α and β.

C. Moments of Z

One approach to finding the moments of Z is to use the foregoing results to find the moment generating function $\mathsf{E}[e^{sZ}]$ and differentiate with respect to s and set $s=0$. Alternatively, one can find moments directly by using the law of total probability. In particular, for the first moment, we have

$$\mathsf{E}[Z] = \bar{\mu}_c(J_c\alpha) + \bar{\mu}_c(J_c(\bar{\mu}(J\beta))), \tag{10}$$

where, for functions $v(\tau)$,

$$\bar{\mu}_c v := \mathsf{E}\bigg[\sum_i v(T_{i0})\bigg], \tag{11}$$

$$(\bar{\mu}v)(t,y) := \mathsf{E}\bigg[\sum_{j\geq 1} v(T_{ij})\bigg| T_{i0}=t, Y_{i0}=y\bigg], \tag{12}$$

$$(J_c\alpha)(t) := \mathsf{E}[\alpha(t,Y_{i0})|T_{i0}=t], \tag{13}$$

and

$$(J\beta)(t,y,\tau) := \mathsf{E}[\beta(\tau,Y_{ij})|T_{i0}=t, Y_{i0}=y, T_{ij}=\tau]. \tag{14}$$

D. More Discussion

In general it is hard to obtain expressions for $\mathbf{L}_c$ and $\widehat{\mathbf{L}}$ that involve only finitely many integrals. Fortunately, it is much easier to find such expressions for $\bar{\mu}_c$ and $\bar{\mu}$, as we will see later when we discuss specific multipath channel examples. For problems related to the power-delay profile and delay spread, simple integral formulas for K_c and K are usually apparent, while J_c and J are often given in closed form by the channel model specification.

We are most interested in the case for which $\alpha(\tau,y)$ and $\beta(\tau,y)$ are both set equal to $\zeta(\tau)g^2$, where $\zeta(\tau)$ is either τ^n or $I_{[0,t]}(\tau)$. Then (4) becomes

$$(K_c\alpha)(\tau) = \mathsf{E}\big[e^{\zeta(\tau)G_{i0}^2}\big|T_{i0}=\tau\big],$$

and (8) becomes

$$(K\beta)(\tau,y,\tau') = \mathsf{E}\big[e^{\zeta(\tau')G_{ij}^2}\big|T_{i0}=\tau, Y_{i0}=y, T_{ij}=\tau'\big].$$

This expresses both operators in terms of the conditional moment generating function of G_{ij}. Similarly, (13) and (14) become

$$(J_c\alpha)(\tau) := \zeta(\tau)\mathsf{E}[G_{i0}^2|T_{i0}=\tau], \tag{15}$$

and

$$(J\beta)(\tau,y,\tau') := \zeta(\tau)\mathsf{E}[G_{ij}^2|T_{i0}=\tau, Y_{i0}=y, T_{ij}=\tau']. \tag{16}$$

This expresses J_c and J in terms of the conditional second moments of the gains, which are usually specified in closed form by the model.

E. LOS vs. NLOS

Among the formulas (3)–(8), the distinction between LOS and NLOS appears only in (3). To see the distinction, write

$$\mathbf{L}_c v = \mathsf{E}\big[e^{v(T_{00})+\sum_{i\geq 1}v(T_{i0})}\big] = e^{v(0)}\mathsf{E}\big[e^{\sum_{i\geq 1}v(T_{i0})}\big]. \tag{17}$$

Hence, in the NLOS case, the factor $e^{v(0)}$ is omitted, while in the LOS case it is retained.

Among the formulas (10)–(14), the distinction between LOS and NLOS appears only in (11). To see the distinction, write

$$\bar{\mu}_c v = \mathsf{E}\bigg[v(T_{00}) + \sum_{i\geq 1} v(T_{i0})\bigg] = v(0) + \mathsf{E}\bigg[\sum_{i\geq 1} v(T_{i0})\bigg]. \tag{18}$$

In the NLOS case, the term $v(0)$ is omitted, while in the LOS case it is retained.

PROBLEMS

1) Derive (2).
2) Derive (10).
3) Derive (9).

REFERENCES

[1] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*. New York: Springer, 1988.

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[3] D. L. Snyder and M. I. Miller, *Random Point Processes in Time and Space*, 2nd ed. New York: Springer, 1991.