Time-Hopped, Direct Sequence Pulses

Let $p(t)$ be a given, deterministic pulse, and define the random waveforms

controls pos. in nth frame

$$\xi_k(t) := \frac{1}{\sqrt{N_s}} \sum_{n=0}^{N_s-1} d_{k,n}\, p(t - nT_f - c_{k,n}T_c),$$

where the $d_{k,n}$ are i.i.d. ± 1 equally likely, the $c_{k,n}$ are i.i.d. $\{0,\ldots,N_h-1\}$-valued, with $P(c_{k,n}=\ell) = \frac{1}{N_h}$ for $\ell = 0,\ldots,N_h-1$. The $\{d_{k,n}\}$ and $\{c_{k,n}\}$ are independent. The $\{d_{k,n}\}_{n=0}^{N_s-1}$ are <u>spreading sequences</u>, and the $\{c_{k,n}\}_{n=0}^{N_s-1}$ are <u>time-hopping sequences</u>.

The Fourier transform of such a waveform is

$$\begin{aligned}
\hat{\xi}_k(f) &:= \int \xi_k(t)\, e^{-j2\pi f t}\, dt \\
&= \frac{1}{\sqrt{N_s}} \sum_{n=0}^{N_s-1} d_{k,n}\, \hat{p}(f)\, e^{-j2\pi f(nT_f + c_{k,n}T_c)} \\
&= \frac{\hat{p}(f)}{\sqrt{N_s}} \sum_{n=0}^{N_s-1} d_{k,n}\, e^{-j2\pi f(nT_f + c_{k,n}T_c)}
\end{aligned}$$

We will need

$$E\Big[\hat{\xi}_k(f_1)\overline{\hat{\xi}_k(f_2)}\Big] = \frac{\hat{p}(f_1)\overline{\hat{p}(f_2)}}{N_s} \sum_{n=0}^{N_s-1}\sum_{m=0}^{N_s-1} E\Big[d_{k,n} d_{k,m} \cdot e^{-j2\pi f_1(nT_f + c_{k,n}T_c)}\, e^{j2\pi f_2(mT_f + c_{k,m}T_c)}\Big]$$

$$= \frac{\hat{p}(f_1)\overline{\hat{p}(f_2)}}{N_s} \sum_{n=0}^{N_s-1}\sum_{m=0}^{N_s-1} E[d_{k,n} d_{k,m}] E\left[e^{-j2\pi f_1 (nT_f + c_{k,n} T_c)} \cdot e^{j2\pi f_2 (mT_f + c_{k,m} T_c)} \right]$$

$$= \frac{\hat{p}(f_1)\overline{\hat{p}(f_2)}}{N_s} \sum_{n=0}^{N_s-1} E\left[e^{-j2\pi (f_1 - f_2)(nT_f)} e^{-j2\pi (f_1-f_2) T_c c_{k,n}} \right]$$

$$= \frac{\hat{p}(f_1)\overline{\hat{p}(f_2)}}{N_s} \sum_{n=0}^{N_s-1} e^{-j2\pi(f_1-f_2)nT_f} \underbrace{\left(\frac{1}{N_h} \sum_{\ell=0}^{N_h-1} e^{-j2\pi(f_1-f_2)T_c \ell} \right)}_{=: A_{N_h, T_c}(f_1 - f_2),}$$

does NOT depend on n.

where
$$A_{N,T}(f) := \frac{1}{N} \sum_{n=0}^{N-1} e^{-j2\pi f T n} = e^{-j\pi(N-1)Tf} \frac{\operatorname{sinc}(NTf)}{\operatorname{sinc}(Tf)}.$$

So,
$$E[\hat{\xi}_k(f_1)\overline{\hat{\xi}_k(f_2)}] = \hat{p}(f_1)\overline{\hat{p}(f_2)}\, A_{N_s T_f}(f_1-f_2) \cdot A_{N_h, T_c}(f_1 - f_2).$$

does NOT depend on k.

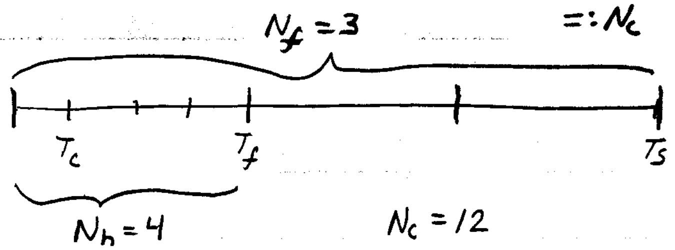
<u>Notes:</u> T_c = chip duration

T_f = frame duration $= N_h T_c \Rightarrow N_h = \#\text{chips/frame}$

T_s = symbol duration $= N_s T_f \Rightarrow N_s = \#\text{frames/symbol}$

$= N_s(N_h T_c)$

$= \underbrace{N_s N_h}_{=: N_c} T_c \Rightarrow N_c = \#\text{chips/symbol}$



If $p(t)$ has duration $\le T_c$, then $\xi_k(t)$ has duration $\le T_s$

Note that since $N_h T_c = T_f$ + $N_c T_c = N_s N_h T_c = N_s T_f$

$$\left| A_{N_s,T_f}(f)\, A_{N_h,T_c}(f) \right|^2 = \frac{\operatorname{sinc}^2(N_s T_f f)}{\operatorname{sinc}^2(T_f f)} \cdot \frac{\operatorname{sinc}^2(N_h T_c f)}{\operatorname{sinc}^2(T_c f)}$$

$$= \frac{1}{N_s^2 N_h^2} \; \frac{\sin^2(\pi \overbrace{N_s T_f}^{= N_c T_c} f)}{\sin^2(\pi T_f f)} \cdot \frac{\sin^2(\pi \overbrace{N_h T_c}^{= T_f} f)}{\sin^2(T_c f)}$$

$$= \frac{1}{N_c^2} \; \frac{\sin^2(\pi N_c T_c f)}{\sin^2(T_c f)} = \frac{\operatorname{sinc}^2(N_c T_c f)}{\operatorname{sinc}^2(T_c f)}$$

$$\longleftrightarrow \frac{1}{N_c} \sum_{n=-\infty}^{\infty} \Delta_n^{N_c} \cdot \delta(t - nT_c).$$

Proof.

$$\int \left(\frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \delta(t-(n-m)T) \right) e^{-j2\pi f t}\, dt = \frac{1}{N^2} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} e^{-j2\pi f(n-m)T}$$

$$= \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{-j2\pi f n T} \right|^2$$

$$= \left| A_{N,T}(f) \right|^2 = \frac{\operatorname{sinc}^2(NTf)}{\operatorname{sinc}^2(Tf)}$$

Also, the above double sum on the left is the sum of the entries in the matrix

$$\begin{bmatrix} \delta(t) & \delta(t+T) & \delta(t+2T) & \cdots & \delta(t+(N-1)T) \\ \delta(t-1T) & \delta(t) & \delta(t+T) & \cdots & \\ \vdots & & \ddots & & \\ \delta(t-(N-1)T) & & & & \end{bmatrix}$$

i.e., $N\delta(t) + (N-1)\delta(t-T) + (N-2)\delta(t-2T) + \cdots$
$+ (N-1)\delta(t+T) + (N-2)\delta(t+2T) + \cdots$

So, the desired time function is

$$\frac{1}{N}\Big\{ \delta(t) + \left(1-\frac{1}{N}\right)\delta(t-T) + \left(1-\frac{2}{N}\right)\delta(t-2T) + \cdots$$
$$+ \left(1-\frac{1}{N}\right)\delta(t+T) + \cdots \Big].$$

$$= \frac{1}{N}\sum_{n=-N}^{N}\left(1-\frac{n}{N}T\right)\delta(t-nT) = \frac{1}{N}\sum_{n=-\infty}^{\infty}\Delta_n^N\cdot\delta(t-nT),$$

where $\Delta_n^N := \begin{cases} 1-n/N, & |n|\le N, \\ 0, & |n|>N. \end{cases}$

We conclude that

$$\left| E\left[\hat{\xi}_k(f_1)\,\overline{\hat{\xi}_k(f_2)}\right]\right|^2 = |\hat{p}(f_1)|^2\,|\hat{p}(f_2)|^2\,\frac{\operatorname{sinc}^2(N_c T_c(f_1-f_2))}{\operatorname{sinc}^2(T_c(f_1-f_2))}.$$

ISI Analysis

Let $\mathcal{C}$ be a constellation of M-1 points in the complex plane $\mathbb{C}$. Let $\{X_n\}_{n=-\infty}^{\infty}$ be i.i.d., $\mathcal{C}$-valued RVs $E[X_n]=0$ and $E[X_n^2]=\theta^2$. We send

$$\sum_{n=-\infty}^{\infty} X_n \xi_n(t-nT_s)$$

over a multipath channel with impulse response

$$h(t) = \sum_i G_i \delta(t-T_i).$$

The waveform seen at the receiver is

$$Y(t) = \sum_{n=-\infty}^{\infty} X_n \Xi_n(t-nT_s) + W(t),$$

$$= X_k \Xi_k(t-kT_s) + \sum_{n\neq k} X_n \Xi_n(t-nT_s) + W(t),$$

where

$$\Xi_n(t) := (h * \xi_n)(t) = \int h(\tau)\xi_n(t-\tau)\, d\tau$$

$$= \sum_i G_i \int \delta(\tau-T_i)\xi_n(t-\tau)\, d\tau$$

$$= \sum_i G_i \xi_n(t-T_i),$$

and $W(t)$ is AWGN with zero mean and $E[W(t)\overline{W(\tau)}] = \sigma^2\delta(t-\tau)$.

Note that $\hat{\Xi}_n(f) = \sum_i G_i \hat{\xi}_n(f) e^{-j2\pi f T_i} = \hat{h}(f)\hat{\xi}_n(f)$, where $\hat{h}(f) := \sum_i G_i e^{-j2\pi f T_i}$.

To decode the kth symbol, we use

$$\langle Y(\cdot + kT_s), \Xi_k \rangle = X_k \|\Xi_k\|^2 + V_k + Z_k,$$

where
$$Z_k := \int W(t+kT_s)\overline{\Xi_k(t)}\,dt \sim N(0,\sigma^2\|\Xi_k\|^2),$$
and
$$V_k := \sum_{n\neq k} X_k \underbrace{\int \Xi_n(t+(k-n)T_s)\overline{\Xi_k(t)}\,dt}_{R_{nk}((k-n)T_s)}$$

Note that
$$R_{nk}(\tau) := \int \Xi_n(t+\tau)\overline{\Xi_k(t)}\,dt \leftrightarrow \hat{\Xi}_n(f)\overline{\hat{\Xi}_k(f)}.$$

Also,
$$E[|R_{nk}(\tau)|^2] = E\Big[\int \hat{\Xi}_n(f_1)\overline{\hat{\Xi}_k(f_1)}\,e^{j2\pi f_1\tau}\,df_1 \cdot \overline{\int \hat{\Xi}_n(f_2)\overline{\hat{\Xi}_k(f_2)}\,e^{j2\pi f_2\tau}\,df_2}\Big]$$
$$= \iint e^{j2\pi(f_1-f_2)\tau} E[\hat{\Xi}_n(f_1)\overline{\hat{\Xi}_n(f_2)}]E[\overline{\hat{\Xi}_k(f_1)\overline{\hat{\Xi}_k(f_2)}}]\,df_1 df_2.$$
$$= \iint e^{j2\pi(f_1-f_2)\tau} E[\hat{h}(f_1)\hat{\xi}_n(f_1)\overline{\hat{h}(f_2)\hat{\xi}_n(f_2)}] \cdot \overline{E[\hat{h}(f_1)\hat{\xi}_k(f_1)\overline{\hat{h}(f_2)\hat{\xi}_k(f_2)}]}\,df_1 df_2$$
$$= \iint e^{j2\pi(f_1-f_2)\tau} |\hat{h}(f_1)|^2|\hat{h}(f_2)|^2|\hat{p}(f_1)|^2|\hat{p}(f_2)|^2 \frac{\operatorname{sinc}^2(N_cT_c(f_1-f_2))}{\operatorname{sinc}^2(T_c(f_1-f_2))}\,df_1 df_2$$

does not depend on n or k.

$$= \iint e^{j2\pi(f_1-f_2)\tau}\hat{q}(f_1)\hat{q}(f_2)D_{N_c}(T_c(f_1-f_2))\,df_1 df_2, \quad (*)$$

where $\hat{q}(f) := |\hat{h}(f)\hat{p}(f)|^2$ & $D_N(f) := \dfrac{\operatorname{sinc}^2(Nf)}{\operatorname{sinc}^2(f)}$

Let us now compute

$$E[|V_k|^2] = E\left[\left(\sum_{n\neq k} X_n R_{nk}((k-n)T_s)\right)\overline{\left(\sum_{m\neq k} X_m R_{mk}((k-m)T_s)\right)}\right]$$

$$= \sum_{n\neq k}\sum_{m\neq k} E[X_n \overline{X_m} R_{nk}((k-n)T_s)\overline{R_{m,k}((k-m)T_s)}]$$

$$= \theta^2 \sum_{n\neq k} E[|R_{nk}((k-n)T_s)|^2]$$

$$= \theta^2 \left\{ \sum_n E[|R_{nk}((k-n)T_s)|^2] - E[|R_{kk}(0)|^2] \right\}$$

Now,

$$\sum_n E[|R_{nk}((k-n)T_s)|^2] = \sum_m E[|R_{nk}(mT_s)|^2]$$

$$= \sum_m \iint e^{j2\pi(f_1-f_2)mT_s}\, \hat{q}(f_1)\hat{q}(f_2)\, D_{N_c}(T_c(f_1-f_2))\, df_1 df_2$$

$$= \iint \left[\frac{1}{T_s}\sum_m \delta((f_1-f_2)-m/T_s)\right]\hat{q}(f_1)\hat{q}(f_2)\, D_{N_c}(T_c(f_1-f_2))\, df_1 df_2$$

$$= \sum_m \frac{1}{T_s}\int \hat{q}(f_2+m/T_s)\,\hat{q}(f_2)\,\underbrace{D_{N_c}(\underbrace{\cancel{T_c}(m/\cancel{T_s})}_{\frac{T_c m}{N_c T_c} = m/N_c})}_{=0 \text{ except when } m=\ell N_c}\, df_2$$

$$= \sum_\ell \frac{1}{T_s}\int \hat{q}\left(f_2+\ell\frac{N_c}{T_s}\right)\hat{q}(f_2)\, df_2$$

$$= \frac{1}{T_s}\sum_\ell \int \hat{q}(f+\ell/T_c)\,\hat{q}(f)\, df$$

$$= \frac{1}{T_S} \sum_{\ell} \int \overline{q(t) e^{-j2\pi(\ell/T_c)t}}\, q(t)\, dt \text{, by Parseval}$$

$$= \frac{1}{T_S} \int |q(t)|^2 \cdot T_c \sum_{\ell} \delta(t - \ell T_c)\; dt$$

$$= \frac{T_c}{T_S} \sum_{\ell} |q(\ell T_c)|^2$$

$$= \frac{1}{N_c} \sum_{\ell} |q(\ell T_c)|^2 .$$

We also have from ⊛ on p. 6 that

$$E[\,|R_{hh}(0)|^2\,] = \int \hat{q}(f_1) \left[\int \hat{q}(f_2)\, D_{N_c}(T_c(f_1 - f_2))\, df_2 \right] df_1$$

$$= \int \overline{q(t)}\; q(t) \frac{1}{N_c} \sum_n \Delta_n^{N_c} \cdot \delta(t - nT_c)\, dt$$

$$= \frac{1}{N_c} \sum_n \Delta_n^{N_c}\, |q(nT_c)|^2 .$$

Thus,

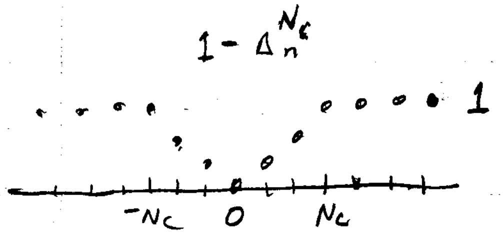
$$E[|V_k|^2] = \theta^2 \cdot \frac{1}{N_c} \sum_n (1 - \Delta_n^{N_c})\, |q(nT_c)|^2 ,$$

$N_c T_c = T_s$ = duration of $\xi_k(t)$

where $q(t) \longleftrightarrow |\hat{h}(f)\hat{p}(f)|^2$

channel spectrum

pulse spectrum $BW \sim \frac{1}{T_c}$



Ref. Lorenzo Piazzo, Tech Rep, "Performance Analysis of DS-IR and DS-UWB in a Frequency selective channel," June 2007 http://infocom.uniroma1.it/~lorenz/trep-01-04-07.pdf