ECE901
HW # 1

1) We write $X \sim \exp(\lambda)$ if the <u>probability density function</u> (<u>pdf</u>) of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The <u>moment generating function</u> (<u>mgf</u>) of X is

$$M(s) := E[e^{sX}].$$

If $X \sim \exp(\lambda)$, show that $M(s) = \dfrac{\lambda}{\lambda - s}$.

2) We say that $X_1, X_2, \ldots$ are <u>i.i.d.</u> if they are <u>independent ($\perp\!\!\!\perp$)</u> and <u>identically distributed</u>; i.e., if they are $\perp\!\!\!\perp$ and all have the same pdf. We say that $T \sim \text{Erlang}(n, \lambda)$ if its pdf is

$$f(t) = \begin{cases} \dfrac{\lambda(\lambda t)^{n-1} e^{-\lambda t}}{(n-1)!}, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

(a) Show that the mgf of T is $[\lambda/(\lambda - s)]^n$.
<u>Hint:</u> The change of variable $\theta = t(\lambda - s)$, $d\theta = (\lambda - s)\,dt$ may be helpful.

(b) If $X_1, \ldots, X_n$ are i.i.d. $\exp(\lambda)$, show that the mgf of $X_1 + \cdots + X_n$ is $[\lambda/(\lambda - s)]^n$.

For $\lambda = 1 \Rightarrow$ (c) Show that $P(T > t) = \sum_{k=1}^{n} t^{k-1} e^{-t}/(k-1)!$. <u>Hint:</u> Use induction on n.

3) We say that a random variable (RV) $U \sim \text{uniform}(0,1)$ if its pdf is

$$f(u) = \begin{cases} 1, & 0 < u < 1, \\ 0, & \text{otherwise}. \end{cases}$$

If $U \sim \text{uniform}(0,1)$, show that $X := (-\ln U)/\lambda$ is an $\exp(\lambda)$ RV. *Hint:* It suffices to show that the cumulative distribution function (cdf) of X is $1-e^{-\lambda x}$ for $x>0$, i.e., show that

$$F_X(x) := \mathbb{P}(X \le x) = 1 - e^{-\lambda x}, \quad x > 0.$$

4) We write $N \sim \text{Poisson}(\lambda)$ if N is a discrete RV with probability mass function (pmf)

$$\mathbb{P}(N=n) = \frac{\lambda^n e^{-\lambda}}{n!}, \quad n = 0, 1, \ldots$$

The probability generating function (pgf) of N is defined as

$$G_N(z) := E[z^N] = \sum_{n=0}^{\infty} z^n \mathbb{P}(N=n)$$

Of course, $M_N(s) = E[e^{sN}] = G_N(e^s)$.
Show that

$$G_N(z) = e^{\lambda(z-1)}.$$

Hint: Recall that the power series for e^x is

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

for all real or complex x.

5) Use the facts: $G_N'(z)\big|_{z=1} = E[N]$ and $G_N''(z)\big|_{z=1} = E[N(N-1)]$ to find the mean and variance of N.
Recall that $\text{var}(N) = E[N^2] - (E[N])^2$.