An Introduction to Certificates of Deposit, Bonds, Yield to Maturity, Accrued Interest, and Duration

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Abstract

A brief introduction is given to compound interest, certificates of deposit, and bonds. The focus is on determining a fair price, yield to maturity, accrued interest, and duration. MATLAB code is given to compute the accrued interest with the 30/360 US method, which is used for US corporate bonds and many US agency bonds.

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

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1. Compound Interest

Recall that if you invest **principal** A_0 at **annual interest rate** r (as a decimal[1]) compounded m times per year for y years, then the amount of money you will have after y years is

$$A(y) = A_0(1+r/m)^{my}.$$

In this formula, m is called the **compounding frequency** and has units of years^{-1}. The reciprocal $1/m$ is called the **compounding period** and has units of years. Since y is measured in years, the product my has no units. The annual interest rate r also has units of years^{-1} so that the quotient r/m has no units.

To compute the amount of money you will have at the time you receive the kth interest payment, set $y = k/m$ for $k = 1, 2, \ldots$. This results in

$$A(k/m) = A_0(1+r/m)^k.$$

For example, if interest is compounded quarterly, then when you receive the first interest payment, after three months ($1/4$ of a year), you will have

$$A(1/4) = A_0(1+r/4).$$

After six months, you will have

$$A(2/4) = A_0(1+r/4)^2.$$

After nine months, you will have

$$A(3/4) = A_0(1+r/4)^3.$$

Since interest is paid at times which are multiples of $1/m$, if y is a time between payment dates, say

$$\frac{k}{m} \le y < \frac{k+1}{m},$$

then we can express y in the form

$$y = \frac{k+\rho}{m}, \quad 0 \le \rho < 1,$$

where ρ is the fraction of a compounding period that has passed since the kth interest payment. With this notation, the amount of money you will have at time $(k+\rho)/m$ is

$$A\left(\frac{k+\rho}{m}\right) = A_0(1+r/m)^{k+\rho}.$$

[1] For example, a 5% annual rate would use $r = 0.05$.

1.1. Daily Compounding

For **daily compounding**, it is more convenient to measure time in days (see [6] for how to do this). After d days you will have

$$A(d) = A_0(1 + r/365)^d.$$

Interest earned using this formula is called **exact interest** to distinguish it from **ordinary interest** which arises using the **banker's rule**,

$$A(d) = A_0(1 + r/360)^d.$$

When bankers talk about interest, they mean ordinary interest unless explicitly stated otherwise.

2. Present Value

Suppose that at some time $y = (k+\rho)/m$ in the future, you will receive an amount of money $A(y)$. How much is it worth today? The answer is called the **present value**, and it is given by

$$\mathrm{PV} = \frac{A\left(\dfrac{k+\rho}{m}\right)}{(1+r/m)^{k+\rho}}.$$

Of course, when $A((k+\rho)/m)$ is given by $A_0(1+r/m)^{k+\rho}$ the present value is simply A_0.

3. Certificates of Deposit

Consider a bank **certificate of deposit (CD)** in which you invest principal A_0 at annual interest rate r paid m times a year. Rather than take the interest payments and spend them, you choose to have your interest added to your CD balance so that you get the benefit of compounding. Suppose that your CD will mature after n/m years so that there will be n compoundings. At maturity your CD will be worth

$$A_0(1 + r/m)^n. \tag{1}$$

However, if you want the current CD balance before maturity, you will have to pay a penalty.

Now suppose that you invest in your CD today at 11 am, and as your turn to leave the bank, they announce that starting at noon, new CDs will earn annual interest rate r_{new}. You return to the bank at noon and strike up a conversation with a potential new

CD customer waiting in line. You ask her how much she will pay you in exchange for your CD that pays the old interest rate r. If $r_{\text{new}} > r$, she will pay you less than A_0, since otherwise she can just buy a new CD directly from the bank. But what is the fair price you should ask for your CD? A little thought suggests that the fair price is $p(r_{\text{new}})$, where $p(r_{\text{new}})$ is chosen so that if she invested $p(r_{\text{new}})$ in a new bank CD at the new rate r_{new}, the value at maturity would equal that of your CD; i.e., $p(r_{\text{new}})$ should solve

$$p(r_{\text{new}})(1+r_{\text{new}}/m)^n = A_0(1+r/m)^n. \tag{2}$$

We conclude that the price should be

$$p(r_{\text{new}}) = \frac{A_0(1+r/m)^n}{(1+r_{\text{new}}/m)^n}.$$

As expected, if $r_{\text{new}} > r$, $p(r_{\text{new}}) < A_0$, and if $r_{\text{new}} < r$, then $p(r_{\text{new}}) > A_0$. ***This illustrates the fact that CD prices and interest rates move in opposite directions.***

As r_{new} ranges over the interval $(-m,\infty)$, the price $p(r_{\text{new}})$ decreases continuously from ∞ to 0. Hence, every positive price corresponds to a unique value of r_{new}. This means that if we know the maturity value of the CD, the number of compoundings n, and the current price, say $\widehat{p}$, we can solve the equation for the current CD interest rate r_{new}; i.e.,

$$r_{\text{new}} = m\left[\left(\frac{\text{maturity\_value}}{\widehat{p}}\right)^{1/n} - 1\right].$$

4. Bond Prices

Consider a **bond** with **face value** F (also called the **maturity value** or **par value**) and annual interest rate r (called the **coupon**, **coupon rate** or **nominal yield**), with coupons paid m times a year. The amount of each interest payment, or **coupon payment**, is

$$\boxed{C := Fr/m.} \tag{3}$$

4.1. A Special Case

Although it is not possible in practice, assume that you will deposit each coupon payment in a savings account that pays annual rate r_{new} compounded m times per year.[2] How much money will you have if the bond matures upon receipt of the nth

[2] It would make more sense to assume that each coupon payment is invested in a bank CD at rate r_{new} that matures when the bond matures. Since successive coupon payments will be invested for shorter and shorter times, the successive values of r_{new} should decrease. However, since we keep r_{new} constant in our analysis, it is simpler to say that the coupon payments are deposited in a savings account.

coupon payment? When the bond matures, you get the face value F plus you have your savings account, whose value is

$$C(1+r_{\text{new}}/m)^{n-1}+C(1+r_{\text{new}}/m)^{n-2}+\cdots+C(1+r_{\text{new}}/m)^{0},$$

where the first term is the result of depositing the first coupon payment in your savings account for the remaining $n-1$ time compounding periods, and the last term is simply the final coupon payment, which spends zero time in your savings account. Hence, at maturity, you have[3]

$$\begin{aligned}A &= F+C(1+r_{\text{new}}/m)^{n-1}+C(1+r_{\text{new}}/m)^{n-2}+\cdots+C(1+r_{\text{new}}/m)^{0}\\ &= F+C\sum_{\ell=0}^{n-1}(1+r_{\text{new}}/m)^{\ell}. \qquad (4)\end{aligned}$$

If the process just described starts exactly n compounding periods prior to the maturity date, what is a fair price for the bond? Because of our savings account assumption, a potential buyer could either invest $p(r_{\text{new}})$ in a savings account at rate r_{new} compounded m times per year, leaving the interest in the bank to compound, or she could buy the bond for $p(r_{\text{new}})$ and deposit the coupon payments in a savings account at rate r_{new} compounded m times per year. Hence, we must have (cf. (2))

$$p(r_{\text{new}})(1+r_{\text{new}}/m)^{n}=A, \qquad (5)$$

or

$$\begin{aligned}p(r_{\text{new}}) = \frac{A}{(1+r_{\text{new}}/m)^{n}} &= \frac{F}{(1+r_{\text{new}}/m)^{n}}+C\sum_{\ell=0}^{n-1}\frac{(1+r_{\text{new}}/m)^{\ell}}{(1+r_{\text{new}}/m)^{n}}\\ &= \frac{F}{(1+r_{\text{new}}/m)^{n}}+C\sum_{\ell=0}^{n-1}\frac{1}{(1+r_{\text{new}}/m)^{n-\ell}}. \qquad (6)\end{aligned}$$

This shows that bond prices and interest rates move in opposite directions.

Example 1. Consider two bonds with the same face value F. The first bond was issued five years ago with coupon rate r to mature in ten years; hence this bond matures five years from today. The second bond is being issued today with rate r_{new}, matures in five years, and sells at par. Use (6) to find today's price of the first bond if $F=\$100$, $r=2.5\%$, and $r_{\text{new}}=2.0\%$.

Solution. We use the following MATLAB code to compute (6).

[3] The proof of Proposition 2 shows that if $r_{\text{new}}=r$, then (4) simplifies to $A=F(1+r/m)^{n}$, which is the CD maturity value (1) with A_0 replaced by F.

```
F = 100;
r = 2.5/100;
rnew = 2.0/100;
m = 2;
n = 5*m; % five years = 10 coupon payments
C = F*r/m;
theta = 1 + rnew/m;
numeratorvec = [ repmat(C,1,n) F ];
powers = [ 1:n n ];
price = sum(numeratorvec./theta.^powers)
```

We find that the price rounds to \$102.37.

Proposition 2. *If r_{new} is equal to the coupon rate r, then the price* (6) *is equal to the bond face value F.*

Proof. First consider the case $r_{\text{new}} = r = 0$. Then $C = Fr/m = 0$ on account of (3), and then (6) reduces to $p(0) = F$.

It remains to consider the case $r_{\text{new}} = r \neq 0$. Put $\theta := 1 + r/m$ so that (4) becomes

$$A = F + C\sum_{\ell=0}^{n-1} \theta^{\ell}.$$

By the **geometric series**,

$$\sum_{\ell=0}^{n-1} \theta^{\ell} = \frac{1-\theta^n}{1-\theta}, \qquad \theta \neq 1.$$

Then use the fact that $1-\theta = -r/m$. Since $C = Fr/m$, we find that

$$A = F + \frac{Fr}{m} \cdot \frac{1-\theta^n}{-r/m} = F + F(\theta^n - 1) = F\theta^n.$$

We can now write (6) as $p(r)\theta^n = F\theta^n$ and the proposition follows. □

Example 3. In the MATLAB code used to solve Example 1, if you change the third line to `rnew = r;` what value do you obtain for `price`?

4.2. The General Case

Let us repeat the analysis leading to (6), but assume that the starting time is midway between coupon payment dates, say a fraction ρ of the compounding period since the most recent coupon payment, and that there n payments remaining. Then the formula (4) for A is the same, but (5) becomes

$$p(r_{\text{new}})(1+r_{\text{new}}/m)^{n-\rho} = A$$

because the time to maturity is no longer n, but is a little shorter by the fraction ρ of a compounding period. It now follows that

$$\begin{aligned}
p(r_{\text{new}}) &= \frac{F}{(1+r_{\text{new}}/m)^{n-\rho}} + C\sum_{\ell=0}^{n-1} \frac{(1+r_{\text{new}}/m)^{\ell}}{(1+r_{\text{new}}/m)^{n-\rho}} \\
&= \frac{F}{(1+r_{\text{new}}/m)^{n-\rho}} + C\sum_{\ell=0}^{n-1} \frac{1}{(1+r_{\text{new}}/m)^{n-\ell-\rho}} \\
&= \frac{F}{(1+r_{\text{new}}/m)^{n-\rho}} + C\sum_{k=1}^{n} \frac{1}{(1+r_{\text{new}}/m)^{k-\rho}}. \qquad (7)
\end{aligned}$$

Even in this slightly more general situation, bond prices and interest rates still move in opposite directions.

Remark. In the next section, we introduce the **yield to maturity**, which is defined as the solution of (7) for r_{new} when the left-hand side is given. Based on our derivation of (7), it appears that the yield to maturity depends on the assumption that the coupon payments are reinvested at rate r_{new}. However, consider the following view suggested by [5]. The first term on the right in (7) is the **present value** of the face value F received at maturity. The fraction $C/(1+r_{\text{new}}/m)^{k-\rho}$ is the present value of the kth coupon payment; i.e., we can write (7) as

$$\text{PV} = \text{PV}_F + \sum_{k=1}^{n} \text{PV}_{C_k}.$$

Now there is no assumption of reinvesting the coupon payments at rate r_{new}.

5. Yield to Maturity — Part 1

Suppose I own the bond described in the previous section, and I make you the following offer. If you pay me $\widehat{p}$ today, then I will give you my remaining interest payments C when I receive them, and I will give you the face value F at maturity. In this offer, there is no mention of an interest rate, so instead of (7), we consider the equation

$$\widehat{p} = \frac{F}{(1+\lambda/m)^{n-\rho}} + C\sum_{k=1}^{n} \frac{1}{(1+\lambda/m)^{k-\rho}}, \qquad (8)$$

and try to solve it for λ. The solution is called the **yield to maturity**. Observe that the right-hand side (RHS) of the equation as a function of λ is continuous and strictly decreasing on $(-m,\infty)$. Since the RHS tends to infinity as $\lambda \searrow -m$ and the RHS tends to zero as $\lambda \nearrow \infty$, the equation can be solved for any positive, finite value of $\widehat{p}$.

Example 4. In Example 1, we showed that the price of the first bond was \$102.37. Use (8) with $\rho = 0$ to obtain the yield to maturity.

Solution. Using the values of `m`, `numeratorvec`, and `powers` from the solution of Example 1, we add the following MATLAB code.

```
phat = 102.37;
v = @(lambda)sum(numeratorvec./(1+lambda/m).^powers);
g = @(lambda)phat-v(lambda);
YTM = fzero(g,0.5)                % Solve phat = v(lambda)
```

What do you expect `YTM` to be?

5.1. Interpretation

Let λ denote the solution of (8), and multiply (8) by $(1+\lambda/m)^{n-\rho}$; i.e., we reverse the steps that led to (7) but replace $p(r_{\text{new}})$ with $\widehat{p}$ and r_{new} with λ. Then

$$\widehat{p}(1+\lambda/m)^{n-\rho} = F + C\sum_{\ell=0}^{n-1}(1+\lambda/m)^{\ell}.$$

The left-hand side is equal to what you would have if you could invest $\widehat{p}$ in a CD paying rate λ until the bond matures. The right-hand side is equal to what you would have if you bought the bond and could invest the coupon payments at rate λ until the bond matures.

6. Buying Bonds

6.1. The Price

When bonds are offered for sale, the price is quoted as a *percentage* of the face value F that you want to buy. For example, the price might be $P_{\%} = 102.763$, meaning that you pay 102.763% of the face value.[4] If you want to buy this bond with a face value $F = \$15,000$, it will cost you

$$F \times \frac{P_{\%}}{100} = \$15,000 \times \frac{102.763}{100} = \$15,414.45.$$

[4] Equivalently, $P_{\%}$ is the price of a bond with a \$100 face value.

In this case, you pay a **premium** of \$414.45, which is 2.763% of the face value. Similarly, if the bond price is $P_{\%} = 98.425$, the cost of a \$15,000 bond would be

$$F \times \frac{P_{\%}}{100} = \$15,000 \times \frac{98.425}{100} = \$14,763.75.$$

In this case, you obtain the bond at a **discount** of $100 - 98.425 = 1.575\%$.

6.2. The Accrued Interest

If you buy a bond between coupon payments dates, when you get your first coupon payment, only a fraction of it really belongs to you. For this reason, at the time you buy the bond, you pay a total of the bond price *plus* a portion of your first coupon payment. That portion is called **accrued interest**, and is denoted by AI. It is computed by solving the equation

$$\boxed{\frac{\mathrm{AI}}{C} = \rho.} \tag{9}$$

Equivalently,

$$\mathrm{AI} = C\rho.$$

Hence, on account of (3), the accrued interest AI is proportional to the face value F.

7. Yield to Maturity — Part 2

From the discussion in Section 6, we should replace $\widehat{p}$ in (8) with

$$F\frac{P_{\%}}{100} + \mathrm{AI},$$

which we call the **total cost**. This results in the formula

$$F\frac{P_{\%}}{100} + \mathrm{AI} = \frac{F}{(1+\lambda/m)^{n-\rho}} + C\sum_{k=1}^{n}\frac{1}{(1+\lambda/m)^{k-\rho}}.$$

Dividing by F results in

$$\frac{P_{\%}}{100} + \frac{\mathrm{AI}}{F} = \frac{1}{(1+\lambda/m)^{n-\rho}} + \frac{C}{F}\sum_{k=1}^{n}\frac{1}{(1+\lambda/m)^{k-\rho}}.$$

Substituting $C = Fr/m$ from (3) and $\mathrm{AI} = C\rho$ from (9) yields

$$\boxed{\frac{P_{\%}}{100} + \frac{r}{m}\rho = \frac{1}{(1+\lambda/m)^{n-\rho}} + \frac{r}{m}\sum_{k=1}^{n}\frac{1}{(1+\lambda/m)^{k-\rho}}.} \tag{10}$$

The value of λ that solves this equation is the **yield to maturity**, which, as we would expect, does not depend on the face value F of the bond. We call (10) the **yield to maturity equation**. The left-hand side is the **total cost** ***per dollar of face value***.

Caution. Of all the parameters in the yield to maturity equation, the fraction of a compounding period ρ is the most difficult to determine, as explained in Section 9. Fortunately, for a bond with a specific face value F, the seller will provide $P_{\%}$, r, m, and AI. Then you can compute C using (3) and then ρ using (9).

8. Sensitivity and Duration

Let $\varphi(\lambda)$ denote the right-hand side of (10). This function gives the total cost per dollar of face value as a function of the yield to maturity. How much does the cost change if the yield changes from λ to $\lambda+\Delta\lambda$? For small $\Delta\lambda$,

$$\varphi(\lambda+\Delta\lambda)-\varphi(\lambda)\approx\varphi'(\lambda)\Delta\lambda.$$

Since φ is a decreasing function (cf. the discussion below (8)), its derivative is negative. Hence if the yield increases, the cost decreases, and vice verse. What we really want, however, is the percentage change in cost,

$$\frac{\varphi(\lambda+\Delta\lambda)-\varphi(\lambda)}{\varphi(\lambda)}\cdot 100\%\approx\frac{\varphi'(\lambda)}{\varphi(\lambda)}\Delta\lambda\cdot 100\%.$$

To make further progress, we need to compute $\varphi'(\lambda)$. To this end, put

$$\psi_t(\lambda):=\frac{1}{(1+\lambda/m)^t},$$

where we suppress the dependence on m. With this notation,

$$\varphi(\lambda)=(r/m)\sum_{k=1}^{n}\psi_{k-\rho}(\lambda)+\psi_{n-\rho}(\lambda).$$

Now observe that

$$\psi_t'(\lambda)=-\frac{t}{m(1+\lambda/m)}\psi_t(\lambda).$$

It follows that

$$\varphi'(\lambda)=-\frac{1}{1+\lambda/m}\left[(r/m)\sum_{k=1}^{n}\frac{k-\rho}{m}\psi_{k-\rho}(\lambda)+\frac{n-\rho}{m}\psi_{n-\rho}(\lambda)\right].$$

Notice that the fractions $(k-\rho)/m$ have units of years, since m has units of years^{-1}. Hence, $\varphi'(\lambda)$ has units of years. The **sensitivity** is defined as

$$S(\lambda) := \frac{\varphi'(\lambda)}{\varphi(\lambda)} = -\frac{1}{1+\lambda/m} \cdot \frac{(r/m)\sum_{k=1}^{n} \frac{k-\rho}{m}\psi_{k-\rho}(\lambda) + \frac{n-\rho}{m}\psi_{n-\rho}(\lambda)}{(r/m)\sum_{k=1}^{n} \psi_{k-\rho}(\lambda) + \psi_{n-\rho}(\lambda)},$$

and has units of years. The **Macaulay duration**, or simply the **duration**, is

$$D(\lambda) := -\frac{(r/m)\sum_{k=1}^{n} \frac{k-\rho}{m}\psi_{k-\rho}(\lambda) + \frac{n-\rho}{m}\psi_{n-\rho}(\lambda)}{(r/m)\sum_{k=1}^{n} \psi_{k-\rho}(\lambda) + \psi_{n-\rho}(\lambda)},$$

and also has units of years, since $1+\lambda/m$ has no units. Clearly, $S(\lambda) = D(\lambda)/(1+\lambda/m)$. On account of this, the sensitivity is usually called the ***modified* duration**.

To put the foregoing all together, suppose the current total cost per dollar of face value is $\varphi(\lambda)$, where λ is the current yield to maturity. If the yield changes to $\lambda+\Delta\lambda$, then the percentage cost change will approximately be

$$S(\lambda)\Delta\lambda \cdot 100\% = \frac{D(\lambda)}{1+\lambda/m}\Delta\lambda \cdot 100\%,$$

where

$$D(\lambda) = -\frac{(r/m)\sum_{k=1}^{n} \frac{(k-\rho)/m}{(1+\lambda/m)^{k-\rho}} + \frac{(n-\rho)/m}{(1+\lambda/m)^{n-\rho}}}{\varphi(\lambda)}.$$

Example 5. Recall the first bond in Example 1 with coupon rate $r = 2.5\%$ and currently priced at \$102.37. Letting $\lambda = r_{\text{new}} = 2.0\%$ denote the current yield, find the duration, sensitivity, and approximate price change if yields increase back to $r = 2.5\%$.

Solution. Using the values of `m`, `numeratorvec`, and `powers` from the solution of Example 1 and the function `v` from Example 4, we can add the following code to compute the duration, sensitivity, and approximate price change (we take $\rho = 0$).[5]

[5] To understand how the code relates to the formula for $D(\lambda)$, keep in mind that `numeratorvec`, which also occurs in the definition of `v`, contains the factor F. Hence, in the code statement that computes `D`, the common factor F cancels.

```
num2 = numeratorvec.*[ 1:n n ]/m;
lambda = rnew;
Dlambda = r - rnew;
D = -sum(num2./(1+lambda/m).^powers)/v(lambda)
S = D/(1+lambda/m)
S*Dlambda*100
```

We find that the *approximate* price change is -2.34%; i.e., the price would drop by $0.0234 \cdot \$102.37 = \$2.40$, and so the new price would be *approximately* \$99.97. Of course the true new price will be the face value of \$100 since the coupon rate is 2.5%. In other words, the true price change would be \$2.37 rather than the approximate \$2.40.

9. Day by Day, or Thirty Days Hath September

Since it was challenging to find the number of days between two dates before computers were commonplace, the following **30/360 US method** continues to be used to compute accrued interest on US corporate bonds and many US agency bonds [8]. If the previous coupon payment was on $M1/D1/Y1$, and the settlement date is $M2/D2/Y2$ (and neither month is February; to handle this case, see [8]), then the number of days between the previous coupon payment date and the settlement date is approximated by the following MATLAB code.[6]

```
if D2==31 && (D1>=30)
   D2 = 30;
end
if D1==31
   D1 = 30;
end
ApproxNumDays = 360*(Y2-Y1)+30*(M2-M1)+(D2-D1);
```

The method also approximates the number of days in a year by 360, and the ratio

$$\frac{\texttt{ApproxNumDays}}{360},$$

is used to replace ρ/m in all of the above formulas; i.e.,

$$\rho = \frac{\texttt{ApproxNumDays}}{360} m.$$

[6] Notice that `M2-M1` or `D2-D1` may be negative.

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