Closable Linear Operators

John A. Gubner
Department of Electrical and Computer Engineering
University of Wisconsin–Madison

If you find this writeup useful, or if you find typos or mistakes, please let me know at gubner@engr.wisc.edu

1. Definitions

Let X and Y be normed vector spaces, and let D_0 be a subspace of X. We say that a linear operator $A\colon D_0 \to Y$ is **closed** if whenever $x_n \in D_0$ with $x_n \to x$ and $Ax_n \to y$, it follows that $x \in D_0$ and $Ax = y$.

Suppose there is a subspace $D \supset D_0$ and a linear operator $B\colon D \to Y$ that satisfies $Ax = Bx$ for $x \in D_0$. In this case, we say that B is an **extension** of A. If the extension B is closed, then we say that A is **closable**. Of course, to say that B is closed means that whenever $x_n \in D$ with $x_n \to x$ and $Bx_n \to y$, then $x \in D$ and $Bx = y$.

2. A Lemma

Lemma 1. *Suppose that whenever $d_n \in D_0$ converges to zero and $Ad_n \to y$, then $y = 0$. Then A is closable.*

Proof. Consider the set

$$D := \{x \in X : \exists x_n \in D_0 \text{ with } x_n \to x \text{ and } Ax_n \to y\}.$$

First notice that $D_0 \subset D$ and that D is a subspace. Next, for $x \in D$, put $Bx := y = \lim_{n\to\infty} Ax_n$. To show that Bx is uniquely defined, suppose $\widetilde{x}_n \in D_0$ with $\widetilde{x}_n \to x$ and $A\widetilde{x}_n \to z$. We must show that $z = y$. Observe that

$$d_n := x_n - \widetilde{x}_n \to x - x = 0 \quad \text{and} \quad Ad_n = A(x_n - \widetilde{x}_n) = Ax_n - A\widetilde{x}_n \to y - z.$$

By hypothesis, $\lim_{n\to\infty} Ad_n = 0$; hence $y - z = 0$.

Clearly, if $x \in D_0$, we can take $x_n = x$ and $Bx = Ax$. Hence, B is an extension of A. It remains to show that B is closed. Suppose $x_n \in D$ with $x_n \to x$ and $Bx_n \to y$. We must show that $x \in D$ and $Bx = y$. To show $x \in D$, we start by finding a sequence from D_0 that converges to x. Since each $x_n \in D$, there is a sequence $w_k^n \to x_n$ and $Aw_k^n \to Bx_n$. Hence, for all sufficiently large k, depending on n, say $k \geq K_n$, we have

$$\|w_k^n - x_n\| < 1/n \quad \text{and} \quad \|Aw_k^n - Bx_n\| < 1/n.$$

In particular, we may specialize to $k = K_n$ and write

$$\|w^n_{K_n} - x_n\| < 1/n \quad \text{and} \quad \|Aw^n_{K_n} - Bx_n\| < 1/n.$$

We can now write

$$\|w^n_{K_n} - x\| \le \|w^n_{K_n} - x_n\| + \|x_n - x\| < 1/n + \|x_n - x\|$$

and

$$\|Aw^n_{K_n} - y\| \le \|Aw^n_{K_n} - Bx_n\| + \|Bx_n - y\| < 1/n + \|Bx_n - y\|.$$

We now see that as $n \to \infty$, $w^n_{K_n} \in D_0$ converges to x and $Aw^n_{K_n}$ converges to y. This says that $x \in D$ and $Bx = y$. □

References

[1] M. Loss, "About closed operators." [Online]. Available: http://people.math.gatech.edu/~loss/13Springtea/closedoperators.pdf, accessed Oct. 12, 2014.