Derivation of the Fourier Inversion Formula, Bochner's Theorem, and Herglotz's Theorem

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Abstract

In Section 1 the Fourier transform is shown to arise naturally in the study of the response of linear, time-invariant systems to sinusoidal inputs. In Section 2, the Dirac delta function is introduced. Since the delta function cannot exist as an ordinary function, it is approximated in Section 3 using summability kernels. In Section 4, summability kernels are used to prove Fourier inversion theorems. In Section 5, positive-semidefinite functions are introduced, and a weak version of Bochner's theorem is proved. This result is then used to motivate Bochner's theorem itself, which is proved using results from probability theory about characteristic functions. Section 6 derives Herglotz's theorem, which is a version of Bochner's theorem for positive-semidefinite sequences.

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

1. Response of Linear Time-Invariant Systems to Sinusoidal Inputs

The Fourier transform arises naturally when considering the response of a linear, time-invariant system to a sinusoidal input. If the system has **impulse response** h, then the response to the complex sinusoid $x(t) = e^{j\omega t}$ is the convolution

$$\begin{aligned}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau &= \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}\,d\tau \\ &= \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau\right)e^{j\omega t} \\ &= H(\omega)e^{j\omega t},\end{aligned}$$

where

$$H(\omega) := \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau$$

is the **Fourier transform** of h.[1][2] We see then that system response to $e^{j\omega t}$ is proportional to $e^{j\omega t}$. In other words, $H(\omega)$ is an **eigenvalue** and $e^{j\omega t}$ is the corresponding **eigenfunction**.

2. The Delta Function

Recall that the **Dirac delta function** is defined by the two properties

$$\delta(t) = 0, \quad t \neq 0, \tag{1}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1. \tag{2}$$

Since $\delta(t) = 0$ for $t \neq 0$, we see that for any function g,

$$g(t)\delta(t) = g(0)\delta(t), \quad \text{for all } t.$$

Thus,

$$\int_{-\infty}^{\infty} g(t)\delta(t)\,dt = \int_{-\infty}^{\infty} g(0)\delta(t)\,dt = g(0)\int_{-\infty}^{\infty} \delta(t)\,dt = g(0), \quad \text{by (2).}$$

[1] We assume that the system under consideration is **stable**; i.e., its impulse response is **integrable** in the sense that

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty.$$

Since h is integrable and since x is bounded, it is clear that the product $h(\tau)x(t-\tau)$ is an integrable function of τ. Thus, the convolution integral is well defined. Similarly, the product $h(\tau)e^{-j\omega\tau}$ is integrable, and the integral for $H(\omega)$ is well defined. We also note that $H(\omega)$ is bounded since

$$|H(\omega)| \leq \int_{-\infty}^{\infty} \big|h(\tau)e^{-j\omega\tau}\big|\,d\tau = \int_{-\infty}^{\infty} |h(\tau)|\,d\tau < \infty.$$

[2] Given $1 \leq p < \infty$, if $\int_{-\infty}^{\infty} |h(t)|^p\,dt < \infty$, we say $h \in L^p$; hence, an impulse response h is stable if $h \in L^1$. We say h has finite energy if $h \in L^2$.

It now follows that the Fourier transform of δ is

$$\Delta(\omega) := \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt = e^{-j\omega\cdot 0} = 1.$$

3. Summability Kernels

Since the delta function cannot exist as an ordinary function, we approximate it with a **summability kernel**. We say that a family of functions $\{\delta_n(t)\}$ is a summability kernel if each δ_n is such that

$$\int_{-\infty}^{\infty} \delta_n(t)\, dt = 1; \tag{3}$$

there is a finite constant M with

$$\int_{-\infty}^{\infty} |\delta_n(t)|\, dt \leq M, \quad \text{for all } n, \tag{4}$$

and for all $\nu > 0$,

$$\lim_{n\to\infty} \int_{\{|t|\geq\nu\}} |\delta_n(t)|\, dt = 0. \tag{5}$$

Clearly (3) is equivalent to (2). A little thought shows that (5) is an attempt to approximate (1). If each $\delta_n(t)$ is nonnegative, then (4) is implied by (3) with $M = 1$. In general, (4) is just a technicality.

Since the Fourier transform of the delta function is one, a great way to define $\delta_n(t)$ is by inverse Fourier transforming some $\Delta_n(\omega)$ as in (6) such that $\Delta_n(\omega) \to 1$. For example [2, Sec. 9.7], if

$$\Delta_n(\omega) := e^{-|\omega|/n},$$

then clearly $\Delta_n(\omega) \to 1$ for all ω. It is also easy to calculate $\delta_n(t)$. Since $\Delta_n(\omega)$ is real and even,

$$\begin{aligned}
\delta_n(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Delta_n(\omega) e^{j\omega t}\, d\omega \qquad (6)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Delta_n(\omega) \cos(\omega t)\, d\omega \\
&= \frac{1}{\pi} \int_{0}^{\infty} \Delta_n(\omega) \cos(\omega t)\, d\omega \\
&= \frac{1}{\pi} \int_{0}^{\infty} e^{-\omega/n} \cos(\omega t)\, d\omega \\
&= \operatorname{Re} \frac{1}{\pi} \int_{0}^{\infty} e^{-\omega/n} e^{j\omega t}\, d\omega \\
&= \frac{n/\pi}{1+(nt)^2}.
\end{aligned}$$

We now verify (3)–(5). To establish (3), write

$$\int_{-\infty}^{\infty} \delta_n(t)\,dt = \frac{2n}{\pi}\int_0^{\infty} \frac{1}{1+(nt)^2}\,dt = \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+\theta^2}\,d\theta = \frac{2}{\pi}\tan^{-1}\theta\Big|_0^{\infty} = 1.$$

Next, since δ_n is nonnegative, (4) holds with $M = 1$. Finally, for (5), write

$$\int_{\nu}^{\infty} \frac{n/\pi}{1+(nt)^2}\,dt = \int_{n\nu}^{\infty} \frac{1/\pi}{1+\theta^2}\,d\theta \to 0 \quad \text{as} \quad n \to \infty.$$

We also note that our particular kernel has one additional property. Observe that for $|t| \geq \nu > 0$,

$$\delta_n(t) = \frac{1}{n}\cdot\frac{n^2/\pi}{1+(nt)^2} \leq \frac{1}{n}\cdot\frac{n^2/\pi}{n^2t^2} \leq \frac{1}{n}\cdot\frac{1}{\pi\nu^2}.$$

Hence,

$$|\delta_n(t)| \leq \frac{B_\nu}{n}, \quad \text{for } |t| \geq \nu, \tag{7}$$

where $B_\nu := (\pi\nu^2)^{-1}$.

Theorem 1 (Sifting). *Let δ_n be a summability kernel that also satisfies* (7). *If h is integrable and continuous at t, then*

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} h(t-\tau)\delta_n(\tau)\,d\tau = h(t).$$

Proof. Write

$$e_n := \int_{-\infty}^{\infty} h(t-\tau)\delta_n(\tau)\,d\tau - h(t).$$

By (3) we can write

$$h(t) = \int_{-\infty}^{\infty} h(t)\delta_n(\tau)\,d\tau,$$

and so

$$e_n = \int_{-\infty}^{\infty} [h(t-\tau)-h(t)]\delta_n(\tau)\,d\tau.$$

Since h is continuous at t, for every $\varepsilon > 0$, there is a $\nu > 0$ such that

$$|\tau| < \nu \ \Rightarrow\ |h(t-\tau)-h(t)| < \varepsilon.$$

Now write

$$|e_n| \leq \int_{\{|\tau|<\nu\}} |h(t-\tau)-h(t)|\,|\delta_n(\tau)|\,d\tau + \int_{\{|\tau|\geq\nu\}} |h(t-\tau)-h(t)|\,|\delta_n(\tau)|\,d\tau.$$

Let I_n and J_n denote these two integrals. Then

$$I_n \le \varepsilon \int_{\{|\tau|<\nu\}} |\delta_n(\tau)|\,d\tau \le \varepsilon \int_{-\infty}^{\infty} |\delta_n(\tau)|\,d\tau \le \varepsilon M,$$

by (4). Next, J_n is upper bounded by

$$\int_{\{|\tau|\ge\nu\}} |h(t-\tau)|\,|\delta_n(\tau)|\,d\tau + \int_{\{|\tau|\ge\nu\}} |h(t)|\,|\delta_n(\tau)|\,d\tau. \tag{8}$$

The integral on the right is equal to

$$|h(t)| \int_{\{|\tau|\ge\nu\}} |\delta_n(\tau)|\,d\tau,$$

which goes to zero by (5). Using (7), the integral on the left in (8) is upper bounded by

$$\frac{B_\nu}{n} \int_{\{|\tau|\ge\nu\}} |h(t-\tau)|\,d\tau \le \frac{B_\nu}{n} \int_{-\infty}^{\infty} |h(t-\tau)|\,d\tau = \frac{B_\nu}{n} \int_{-\infty}^{\infty} |h(\theta)|\,d\theta. \qquad \square$$

4. Fourier Inversion Theorems

We begin with the well-known fact that convolution in the time domain corresponds to multiplication in the frequency domain.

Theorem 2 (Convolution). *If h is integrable with Fourier transform H, and if G is integrable with inverse Fourier transform g, then*[3]

$$\int_{-\infty}^{\infty} h(t-\tau)g(\tau)\,d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)G(\omega)e^{j\omega t}\,d\omega.$$

Proof. By a change of variable,

$$\int_{-\infty}^{\infty} h(t-\tau)g(\tau)\,d\tau = \int_{-\infty}^{\infty} h(\theta)g(t-\theta)\,d\theta.$$

Since g is an inverse transform,

$$g(t-\theta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)e^{j\omega(t-\theta)}\,d\omega.$$

[3] Both integrals in the Convolution Theorem are well defined. Since g is the inverse Fourier transform of the integrable function G, g is bounded (recall footnote 1). Since h is integrable so is the product $h(t-\tau)g(\tau)$. Similarly, since H is bounded (by footnote 1) and G is integrable, so is $H(\omega)G(\omega)e^{j\omega t}$.

Substituting this formula shows that the convolution is equal to

$$\int_{-\infty}^{\infty} h(\theta)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega(t-\theta)}\,d\omega\right]d\theta,$$

or[4]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left[\int_{-\infty}^{\infty} h(\theta)e^{-j\omega\theta}\,d\theta\right]e^{j\omega t}\,d\omega,$$

which is the desired result. □

Theorem 3. *Under the conditions of the Sifting Theorem, if δ_n is the inverse transform of some integrable Δ_n, then*

$$h(t) = \lim_{n\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\Delta_n(\omega)e^{j\omega t}\,d\omega. \tag{9}$$

Proof. Use the Sifting and Convolution Theorems. □

Theorem 4 (Inversion). *If h is integrable and continuous at t, and if its transform H is also integrable, then*

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega.$$

Proof. We first apply the previous theorem to a bounded sequence $\Delta_n(\omega) \to 1$. Since $H(\omega)$ is integrable, the dominated convergence theorem allows us to bring the limit inside the integral in (9). The result follows because $\Delta_n(\omega) \to 1$ for all ω. □

Example. Take $h(t) = (1-|t|)I_{[-1,1]}(t)$. Clearly, h is continuous and integrable. Since h is real valued and even, an easy calculation shows that

$$H(\omega) = 2\int_0^1 (1-t)\cos(\omega t)\,dt = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^2.$$

Since H is also integrable, and since h is continuous,

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(\omega/2)}{\omega/2}\right)^2 e^{j\omega t}\,d\omega.$$

In particular, if we take $t=0$ and use the fact that $h(0)=1$, we find that

$$\int_{-\infty}^{\infty}\left(\frac{\sin\omega}{\omega}\right)^2 d\omega = \pi. \tag{10}$$

[4] Changing the order of integration is justified by the Tonelli and Fubini Theorems, and uses the fact that both h and G are integrable.

Another popular summability kernel is obtained by taking

$$\Delta_n(\omega) := \Big(1 - \frac{|\omega|}{n}\Big) I_{[-n,n]}(\omega).$$

Since $\Delta_n(\omega) = h(\omega/n)$, we can use the result of the preceding example to write

$$\delta_n(t) = \frac{n}{2\pi} H(-nt) = \frac{n}{2\pi}\Big(\frac{\sin(nt/2)}{nt/2}\Big)^2.$$

On account of (10), we see that (3) holds. Since δ_n is nonnegative, (4) holds automatically with $M = 1$. It is also easy to see that (5) holds. Property (7) obviously holds with $B_\nu = 2/(\pi\nu^2)$. By Theorem 3, if h is integrable and continuous at t, then

$$h(t) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} \Big(1 - \frac{|\omega|}{n}\Big) H(\omega) e^{j\omega t}\, d\omega. \tag{11}$$

Both of the summability kernels we have introduced are nonnegative and their transforms satisfy

$$0 \le \Delta_n(\omega) \le \Delta_{n+1}(\omega) \le 1.$$

This fact can be used to prove the following useful result.

Theorem 5 (Positive Inversion). *If h is integrable and continuous at $t = 0$, and if its transform H is nonnegative, then H is integrable. For all t at which h is continuous,*

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega.$$

Proof. We can apply Theorem 3 to either of our bounded, nonnegative summability kernels. Taking $t = 0$ in (9) yields

$$h(0) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\Delta_n(\omega)\, d\omega.$$

Since the integrands are nonnegative and nondecreasing, the monotone convergence theorem allows us to bring the limit inside the integral. Hence,

$$h(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)\, d\omega,$$

and we see that $H(\omega)$ is integrable. Then, as argued in the proof of the Inversion Theorem, we can bring the limit inside the integral in (9) for arbitrary continuity points t. □

4.1. The L^2 Theory†

Lemma 6. *If $g \in L^2 \cap L^1$, then its autocorrelation function*

$$h(t) := \int_{-\infty}^{\infty} g(t+\tau)\overline{g(\tau)}\,d\tau$$

is a bounded, integrable, continuous function, and the transform of the autocorrelation function is

$$H(\omega) = |G(\omega)|^2.$$

Proof. Since $g \in L^2$, we can write $h(t) = \langle g(t+\cdot), g\rangle$. Then by the Cauchy–Schwarz inequality, $|h(t)| \le \|g\|_2^2$. To show that $h \in L^1$, write[5]

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} g(t+\tau)\overline{g(\tau)}\,d\tau \right| dt \\ &\le \int_{-\infty}^{\infty} |g(\tau)| \int_{-\infty}^{\infty} |g(t+\tau)|\,dt\,d\tau = \|g\|_1^2.\end{aligned}$$

Since $h \in L^1$, we can write its Fourier transform integral and change the order of integration as above to obtain

$$H(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(t+\tau)\overline{g(\tau)}\,d\tau \right] e^{-j\omega t}\,dt = |G(\omega)|^2.$$

To establish the continuity of h at any point t, suppose that $t_n \to t$. Then $\|g(t_n+\cdot) - g(t+\cdot)\|_2 \to 0$ by continuity of translation [2, p. 196, Th. 9.5]. By continuity of the inner product, $\langle g(t_n+\cdot), g\rangle \to \langle g(t+\cdot), g\rangle$; i.e., $h(t_n) \to h(t)$. Since the sequence t_n was arbitrary, h is continuous at t. □

Lemma 7. *If $g \in L^2 \cap L^1$, then $G \in L^2$ and $\|g\|_2^2 = \frac{\|G\|_2^2}{2\pi}$.*

Proof. Apply the Positive Inversion Theorem with $t = 0$ to the autocorrelation function h of the preceding lemma, and note that $h(0) = \|g\|_2^2$ and $\int_{-\infty}^{\infty} H(\omega)\,d\omega = \int_{-\infty}^{\infty} |G(\omega)|^2\,d\omega = \|G\|_2^2$. □

For arbitrary $g \in L^2$, put $g_n(t) := g(t)I_{[-n,n]}(t)$. Then $g_n \in L^2 \cap L^1$ by Hölder's inequality. Further, since $\|g_n - g\|_2 \to 0$, g_n is Cauchy in L^2. By the preceding lemma, $\|G_n - G_m\|_2 \to 0$ as $n, m \to \infty$; i.e., G_n is Cauchy in L^2. Hence, there exists

† The material in this subsection is not needed for Bochner's Theorem.

[5] Changing the order of integration is justified by Tonelli's Theorem.

a $G \in L^2$ with $\|G_n - G\|_2 \to 0$. This limit G is the **Fourier transform** of $g \in L^2$. Furthermore,[6]

$$\|g\|_2 = \lim_{n\to\infty} \|g_n\|_2 = \frac{1}{\sqrt{2\pi}} \lim_{n\to\infty} \|G_n\|_2 = \frac{1}{\sqrt{2\pi}} \|G\|_2,$$

which is **Parseval's equation**.

5. Bochner's Theorem

A function h is **positive semidefinite** if for any times $t_1, \ldots, t_n$ and any complex constants $c_1, \ldots, c_n$,

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \overline{c_i} h(t_i - t_k) c_k \geq 0. \tag{12}$$

Taking $n = 1$ and $c_1 = 1$ shows that $h(0) \geq 0$.

Lemma 8. *If h is a continuous, positive-semidefinite function, then for $T > 0$,*

$$H_T(\omega) := \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right) h(t) e^{-j\omega t}\, dt \geq 0. \tag{13}$$

Proof. The proof consists of two parts. We first show that for any $T > 0$,

$$\int_{-T}^{T} \int_{-T}^{T} h(t-\theta) e^{-j\omega(t-\theta)}\, dt\, d\theta \geq 0. \tag{14}$$

We then show that this double integral is equal to $2TH_{2T}(\omega)$. It follows that $H_T(\omega) \geq 0$ for all $T > 0$.

Since h is continuous, the integral in (14) is equal to a limit of finite Riemann sums of the form

$$\sum_i \sum_k h(t_i - t_k) e^{-j\omega t_i} \Delta t_i e^{j\omega t_k} \Delta t_k.$$

These sums are nonnegative because h is positive semidefinite. This establishes (14).

It remains to simplify the double integral in (14). Write it as

$$\int_{-T}^{T} \int_{-\infty}^{\infty} I_{[-T,T]}(t) h(t-\theta) e^{-j\omega(t-\theta)}\, dt\, d\theta.$$

Make the change of variable $\tau = t - \theta$ and obtain

$$\int_{-T}^{T} \int_{-\infty}^{\infty} I_{[-T,T]}(\tau+\theta) h(\tau) e^{-j\omega\tau}\, d\tau\, d\theta.$$

[6] By the triangle inequality, $\|x+y\| \leq \|x\| + \|y\|$, it follows that $\big|\|x\| - \|y\|\big| \leq \|x-y\|$. Hence, if $\|x_n - x\| \to 0$, then $\|x_n\| \to \|x\|$.

Now change the order of integration to get

$$\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\left(\int_{-T}^{T} I_{[-T,T]}(\tau+\theta)\,d\theta\right)d\tau. \tag{15}$$

To evaluate the inner integral with respect to θ, make the change of variable $s = -\theta$. Then the inner integral becomes

$$\int_{-T}^{T} I_{[-T,T]}(\tau-s)\,ds = \int_{-\infty}^{\infty} I_{[-T,T]}(s)\,I_{[-T,T]}(\tau-s)\,ds.$$

In other words, we have the convolution of two rectangular pulses. Hence,

$$\int_{-T}^{T} I_{[-T,T]}(\tau+\theta)\,d\theta = \begin{cases} 2T-|\tau|, & |\tau| \le 2T, \\ 0, & \text{otherwise.} \end{cases}$$

Substituting this into (15), which is equal to the double integral in (14), we find that (14) is equal to $2TH_{2T}(\omega)$. □

Theorem 9 (Bochner, weak version). *Let h be an integrable, continuous, positive-semidefinite function. Then*

$$H(\omega) := \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt \ge 0, \tag{16}$$

H is integrable, and

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}\,d\omega. \tag{17}$$

Proof. Since h is integrable, in (13) we can let $T = n$ and apply the dominated convergence theorem to obtain (16). The integrability of H and (17) follow immediately from the Positive Inversion Theorem. □

To motivate Bochner's Theorem, observe that if $h(0) > 0$, then (17) can be rewritten as

$$\frac{h(t)}{h(0)} = \int_{-\infty}^{\infty} e^{j\omega t}\,\frac{H(\omega)}{2\pi h(0)}\,d\omega.$$

The quotient on the right is nonnegative. Taking $t = 0$ shows that it integrates to one, and is therefore a probability density. If we denote the corresponding cumulative distribution function by F, then

$$\frac{h(t)}{h(0)} = \int_{-\infty}^{\infty} e^{j\omega t}\,dF(\omega),$$

or, if $K(\omega) := 2\pi h(0)F(\omega)$,

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\,dK(\omega).$$

Usually K is called the **spectral distribution function**. Bochner's theorem asserts the existence of such a distribution function even if h is not integrable.

Theorem 10 (Bochner). *Let h be a continuous, positive-semidefinite function with $h(0) > 0$. Then there is a cumulative distribution function F such that*

$$\frac{h(t)}{h(0)} = \int_{-\infty}^{\infty} e^{j\omega t}\,dF(\omega), \tag{18}$$

or, writing $K(\omega) := 2\pi h(0)F(\omega)$,

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\,dK(\omega).$$

Proof. Let

$$q_T(t) := \left(1 - \frac{|t|}{T}\right) I_{[-T,T]}(t)$$

so that the integral in (13) can be written as

$$H_T(\omega) = \int_{-\infty}^{\infty} q_T(t)h(t)e^{-j\omega t}\,dt.$$

Since the product $q_T\,h$ is integrable and continuous and has nonnegative transform H_T, the Positive Inversion Theorem tells us that H_T is integrable, and

$$q_T(t)h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_T(\omega)e^{j\omega t}\,d\omega,$$

which we can rewrite as

$$\frac{q_T(t)h(t)}{h(0)} = \int_{-\infty}^{\infty} e^{j\omega t}\,\frac{H_T(\omega)}{2\pi h(0)}\,d\omega.$$

This equation shows that the left-hand side is the characteristic function of the probability density $H_T(\omega)/2\pi h(0)$. Since $q_T(t) \to 1$, the characteristic functions on the left converge to the continuous limit $h(t)/h(0)$. By [1, p. 350, Corollary 1], there exists a limiting cumulative distribution function F to which the cumulative distributions corresponding to $H_T(\omega)/[2\pi h(0)]$ converge, and the characteristic function of F is the limit $h(t)/h(0)$; i.e., (18) holds. $\square$

Remark. From (18) we see that $h(-t) = \overline{h(t)}$ and $|h(t)| \le h(0)$. In fact, these are implied directly by the positive semidefiniteness of h. Let Q denote the $n \times n$ matrix with elements $h(t_i - t_k)$, and let $c = [c_1, \ldots, c_n]'$. Then (12) says that $c^*Qc \ge 0$ for all c. Taking $c = x + \lambda y$ and considering the cases $\lambda = j$ and $\lambda = 1$ shows that $y^*Qx = y^*Q^*x$ for all x and y. This implies that $Q^* = Q$. For the case $n = 2$ with $t_1 = t$ and $t_2 = 0$, $Q = Q^*$ implies $h(-t) = \overline{h(t)}$. Let $c = [e^{j\theta}, e^{j\varphi}]'$. Then $c^*Qc \ge 0$ is equivalent to

$$h(0) \ge -\big[e^{j(\varphi-\theta)}h(t) + e^{-j(\varphi-\theta)}h(-t)\big]/2.$$

Using the fact that $h(-t) = \overline{h(t)}$ and writing $h(t) = |h(t)|e^{j\zeta}$, where $\zeta = \arg h(t)$, we find that

$$\frac{h(0)}{|h(t)|} \ge -\cos(\varphi - \theta + \zeta).$$

Since φ and θ are arbitrary, we must have $h(0)/|h(t)| \ge 1$, or $h(0) \ge |h(t)|$.

6. Herglotz's Theorem — The Discrete Bochner Theorem

If $h(n)$ is a discrete-time positive semidefinite sequence, then we would like to write in analogy with (16) and (17),

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \ge 0, \tag{19}$$

and

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}\,d\omega. \tag{20}$$

The analog of Bochner's Theorem would be

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega n}\,dK(\omega), \tag{21}$$

where $K(\omega) = 2\pi h(0)F(\omega)$ for some cumulative distribution function F.

We begin by noting that an argument analogous to that in the proof of Lemma 8 shows that

$$H_N(\omega) := \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big)h(n)e^{-j\omega n} \ge 0. \tag{22}$$

Hence, if h is summable; i.e., if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty,$$

then $H_N \to H$ in (19) and so H is well defined and nonnegative. Furthermore, the series in (19) is absolutely and uniformly convergent. It follows that H is continuous and therefore integrable on any finite interval. Thus, the integral in (20) is well defined. Substituting the series for H into the integral in (20) and interchanging the order of summation and integration (justified by the uniform convergence) shows that (20) holds.

To prove the discrete analog of Bochner's Theorem is more puzzling because we cannot talk about the continuity of a discrete-time sequence. However, in the proof of Bochner's Theorem, all that the continuity of $h(t)$ really bought us was the **tightness** of the sequence of cumulative distributions corresponding to the densities $H_T(\omega)/2\pi h(0)$ [1, p. 304, Proof of Corollary 1]. Fortunately, the analogous cumulative distributions in the discrete-time problem all live on a finite interval and are therefore always tight.

Theorem 11 (Herglotz). *If $h(n)$ is a positive semidefinite sequence with $h(0) > 0$, then there is a cumulative distribution function F such that if $K(\omega) := 2\pi h(0)F(\omega)$, then* (21) *holds.*

Proof. With H_N defined in (22), it is easy to see that[7]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H_N(\omega)e^{j\omega k}\,d\omega = \Big(1-\frac{|k|}{N+1}\Big)h(k)$$

for $|k| \le N$, and in particular,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H_N(\omega)\,d\omega = h(0).$$

Consider the cumulative distribution function

$$F_N(\omega) := \int_{-\pi}^{\omega} \frac{H_N(\theta)}{2\pi h(0)}\,d\theta, \quad -\pi \le \omega \le \pi.$$

For $\omega < -\pi$, put $F_N(\omega) := 0$ and for $\omega > \pi$, put $F_N(\omega) := 1$. Since the corresponding densities are supported on a common finite interval, the cumulative distributions are tight [1, p. 290], and so there exists a subsequence N_i and a limiting cumulative distribution function F such that for all bounded and continuous functions g [1, pp. 288–289, Theorem 25.8],

$$\lim_{i\to\infty}\int_{-\infty}^{\infty} g(\omega)\,dF_{N_i}(\omega) = \int_{-\infty}^{\infty} g(\omega)\,dF(\omega).$$

[7] Since H_N is defined by a *finite* sum, we do not need a discrete-time version of the Positive Inversion Theorem. Hence, no preliminary results on Fourier series are needed here.

Since $F_{N_i}(\omega) = 0$ for $\omega < -\pi$ and $F_{N_i}(\omega) = 1$ for $\omega > \pi$, the limit $F(\omega)$ inherits these properties. Hence, the above formula reduces to

$$\lim_{i\to\infty} \int_{-\pi}^{\pi} g(\omega)\, dF_{N_i}(\omega) = \int_{-\pi}^{\pi} g(\omega)\, dF(\omega),$$

and we can write

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{j\omega n}\, dF(\omega) &= \lim_{i\to\infty} \int_{-\pi}^{\pi} e^{j\omega n}\, dF_{N_i}(\omega) \\
&= \lim_{i\to\infty} \int_{-\pi}^{\pi} e^{j\omega n} \frac{H_{N_i}(\omega)}{2\pi h(0)}\, d\omega \\
&= \lim_{i\to\infty} \left(1 - \frac{|n|}{N_i + 1}\right) \frac{h(n)}{h(0)} \\
&= \frac{h(n)}{h(0)}.
\end{aligned}$$

Formula (21) now follows. □

References

[1] P. Billingsley, *Probability and Measure*, 3rd ed. New York: Wiley, 1995.

[2] W. Rudin, *Real and Complex Analysis*, 2nd ed. New York: McGraw-Hill, 1974.

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