Introduction to Frequency Analysis and the DFT (Long Version)

John A. Gubner
Department of Electrical and Computer Engineering
University of Wisconsin–Madison

Contents

Abstract

Frequency analysis is introduced starting with the Fourier transform applied to finite-duration waveforms, say T seconds. Then the Fourier series is introduced, and it is pointed out that the Fourier series coefficients are proportional to samples of the Fourier transform taken at frequencies that are integer multiples of $1/T$.

The approximation of the Fourier transform integral by Riemann sums is used to introduce the discrete Fourier transform (DFT). It is shown that if a continuous-time waveform is sampled at rate f_s for N samples, then the DFT of the samples, divided by f_s, is approximately equal to the Fourier transform evaluated at frequencies that are integer multiples of f_s/N. If $f_s = N/T$, then these evaluation frequencies are the multiples of $1/T$ associated with the Fourier series coefficients. If the equation $f_s = N/T$ does not hold, this will not be the case; e.g., zero padding.

To conclude, we derive the sampling theorem for periodic waveforms, and we give conditions under which the DFT approximation of the Fourier transform is exact.

It is assumed that the reader is familiar with **Euler's formulas**

$$e^{j\theta} = \cos\theta + j\theta, \quad \cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}, \quad \text{and} \quad \sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}.$$

To understand the last section, the reader needs to be familiar with the **geometric series**, which says that for any complex number z,

$$\sum_{m=0}^{N-1} z^m = \begin{cases} \dfrac{1-z^N}{1-z}, & z \neq 1, \\ N, & z = 1. \end{cases}$$

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

1. Introduction

The goal of frequency analysis is to process a signal $x(t)$ and determine what frequencies are present. For example, the standard form of a **sinusoid** is

$$x(t) = A\cos(2\pi f_0 t + \varphi),$$

where $A > 0$ is the **amplitude**, f_0 is the **frequency**, measured in units of inverse seconds or Hertz (Hz), and φ is the **phase**, measured in radians.

In practice, we never observe $x(t)$ for all $-\infty < t < \infty$. We typically have $x(t)$ only for t in a finite time interval, say $[0,T]$. And in the digital world, we have even less information. We have only finitely many samples, say N samples. In other words, we have only $x(n\Delta t)$ for $n = 0,\ldots,N-1$, where $f_s := 1/\Delta t$ is called the **sampling rate**. Our goal is to understand how to take finitely many samples of a signal $x(t)$ and determine what frequencies are present. We begin by focusing on the case when we observe $x(t)$ for all $0 \leq t \leq T$. Once we understand how this works, we proceed to the case of finitely many samples.

2. The Sinc Function

The **sinc** function is the key to our understanding. This function is defined by

$$\operatorname{sinc}(\theta) := \begin{cases} \dfrac{\sin(\pi\theta)}{\pi\theta}, & \theta \neq 0, \\ 1, & \theta = 0, \end{cases}$$

and is plotted in Fig. 1. The reason for including the factor of π is to make the zero crossings occur when θ is a nonzero integer. Another important feature of $\operatorname{sinc}(\theta)$ is that the tallest peak occurs at $\theta = 0$, has height 1, and width 2.[a]

[a] The width of a peak is measured between the zeros on either side. The tallest peak in Fig. 1 sits between the zeros at $\theta = \pm 1$, so the width is $(+1)-(-1) = 2$.

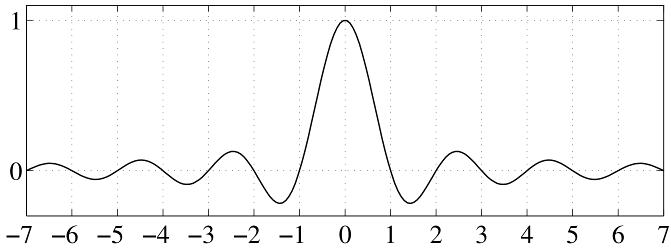

Figure 1. The function $\operatorname{sinc}(\theta)$.

3. The Fourier Transform

Given a signal $x(t)$ defined for $-\infty < t < \infty$, its **Fourier transform** or **spectrum** is

$$X(f) := \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt \tag{1}$$

whenever this integral is defined. The quantity $|X(f)|$ is called the ***magnitude*** **spectrum**.

Even when $x(t)$ is defined for all t, in practice, we can observe it or measure it only for a finite length of time, say $0 \le t \le T$. In this case, we focus on

$$\int_0^T x(t) e^{-j2\pi ft}\, dt. \tag{2}$$

For example, suppose we observe $x(t) = 1$ for $0 \le t \le T$. Then we need to compute $\int_0^T e^{-j2\pi ft}\, dt$.

Example 1. Show that

$$\boxed{P_T(f) := \int_0^T e^{-j2\pi ft}\, dt = e^{-j\pi Tf} \cdot T \cdot \operatorname{sinc}(Tf).} \tag{3}$$

Solution. Taking $f = 0$ in the above integral yields $P_T(0) = \int_0^T 1dt = T$. For $f \neq 0$, we take the antiderivative of $e^{-j2\pi ft}$ and write

$$\begin{aligned} P_T(f) = \frac{e^{-j2\pi ft}}{-j2\pi f}\bigg|_{t=0}^{t=T} &= \frac{e^{-j2\pi fT} - e^{-j2\pi f0}}{-j2\pi f} \\ &= \frac{1 - e^{-j2\pi fT}}{j2\pi f} \end{aligned}$$

$$= e^{-j\pi Tf} \cdot \frac{e^{j\pi Tf} - e^{-j\pi Tf}}{j2\pi f}$$
$$= e^{-j\pi Tf} \cdot T \cdot \frac{\sin(\pi Tf)}{\pi Tf}, \quad \text{by Euler's formula.}$$

The function $P_T(f)$ can be defined in MATLAB with the statement

```
PT = @(f)T*exp(-1j*pi*T*f).*sinc(T*f);
```

Since this function is complex valued, we usually plot its magnitude,

$$|P_T(f)| = T|\operatorname{sinc}(Tf)|$$

as shown in Fig. 2. Since $\operatorname{sinc}(\theta) = 0$ whenever θ is a nonzero integer, we must have $\operatorname{sinc}(Tf) = 0$ whenever $f = m/T$ for a nonzero integer m. Also, the tallest peak has height T and width $2/T$.

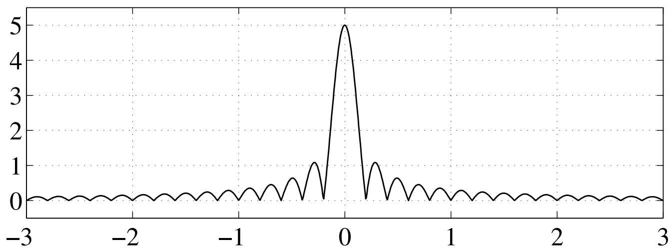

Figure 2. The function $|P_T(f)| = T|\operatorname{sinc}(Tf)|$ for $T = 5$. The height of the tallest peak is $T = 5$ and occurs at $f = 0$.

Example 2. If the signal $A\,e^{j2\pi f_0 t}$ is measured for $0 \le t \le T$, show that its Fourier transform is $A \cdot P_T(f - f_0)$.

Solution. Write

$$\int_0^T \big(A\,e^{j2\pi f_0 t}\big)e^{-j2\pi ft}\,dt = A\int_0^T e^{-j2\pi(f-f_0)t}\,dt = A \cdot P_T(f - f_0),$$

where the second equality follows from (3) with f replaced by $f - f_0$.

Example 3. If a signal having the form[b]

$$\sum_k x_k e^{j2\pi f_k t}$$

[b] Although it may seem restrictive to assume signals of this form, there is no loss of generality as we see later in Section 4.

is measured for $0 \le t \le T$, show that its Fourier transform is

$$\int_0^T \left(\sum_k x_k e^{j2\pi f_k t} \right) e^{-j2\pi f t}\, dt = \sum_k x_k P_T(f - f_k).$$

Solution. Write

$$\begin{aligned}\int_0^T \left(\sum_k x_k e^{j2\pi f_k t} \right) e^{-j2\pi f t}\, dt &= \sum_k x_k \int_0^T e^{j2\pi f_k t} e^{-j2\pi f t}\, dt \\ &= \sum_k x_k \int_0^T e^{-j2\pi (f - f_k) t}\, dt \\ &= \sum_k x_k P_T(f - f_k),\end{aligned}$$

where the last step follows from (3) with f replaced by $f - f_k$.

Example 4. Consider the waveform

$$a e^{j2\pi f_a t} + b e^{j2\pi f_b t} + c e^{j2\pi f_c t},$$

whose ***line*** **spectrum** is shown in Fig. 3. As the plot shows, $f_a < f_b < f_c$. This

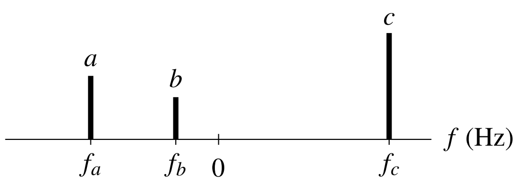

Figure 3. Line spectrum of $a e^{j2\pi f_a t} + b e^{j2\pi f_b t} + c e^{j2\pi f_c t}$.

waveform is observed for $0 \le t \le T = 6$, and its magnitude spectrum is shown in the top graph of Fig. 4. Determine the frequencies f_a, f_b, f_c and the corresponding values of $|a|$, $|b|$, $|c|$.

Solution. Taking the transform term by term as justified by Example 3 , we obtain

$$a P_T(f - f_a) + b P_T(f - f_b) + c P_T(f - f_c).$$

From the graph it appears that $f_a = -1.5$ Hz, $f_b = -0.5$ Hz, and $f_c = 2$ Hz. The corresponding peak heights are 3, 2, and 5. Recalling that the tallest peak of each $P_T(\cdot)$ factor is T, we divide the peak heights by $T = 6$ to get $|a| = 3/6 = 1/2$, $|b| = 2/6 = 1/3$, and $|c| = 5/6$. *Of course, if we had plotted the magnitude spectrum divided by T, we could have read $|a|$, $|b|$, and $|c|$ directly from the graph!*

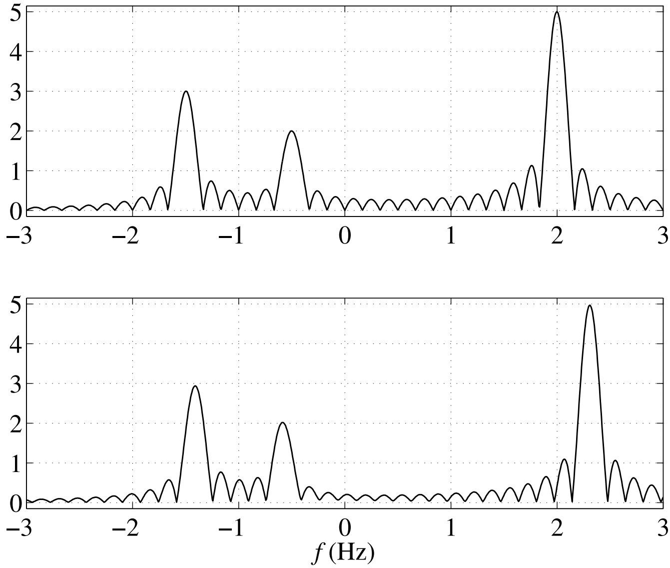

Figure 4. Magnitude spectra of $ae^{j2\pi f_a t} + be^{j2\pi f_b t} + ce^{j2\pi f_c t}$ observed for $0 \le t \le T = 6$. The top and bottom graphs differ only in the values of f_a, f_b, and f_c.

Example 5. If we change the frequencies f_a, f_b, and f_c a little bit, we get the bottom graph in Fig. 4. This graph was generated with $f_a = -\sqrt{2} \approx -1.414$ Hz, $f_b = -1/\sqrt{3} \approx -0.5774$ Hz, and $f_c = \sqrt{5} \approx 2.2361$ Hz.

Example 6. A sinusoid is measured for $0 \le t \le T$. Find the formula for its spectrum and then plot its *line* spectrum.

Solution. Applying Euler's formula to the standard form of a sinusoid, we have

$$A\cos(2\pi f_0 t + \varphi) = A\frac{e^{j(2\pi f_0 t+\varphi)} + e^{-j(2\pi f_0 t+\varphi)}}{2} = \frac{Ae^{j\varphi}}{2}e^{j2\pi f_0 t} + \frac{Ae^{-j\varphi}}{2}e^{j2\pi(-f_0)t}.$$

Using the technique of Example 3, the spectrum is

$$\frac{Ae^{j\varphi}}{2}P_T(f - f_0) + \frac{Ae^{-j\varphi}}{2}P_T(f + f_0). \tag{4}$$

The corresponding line spectrum is shown in Fig. 5.

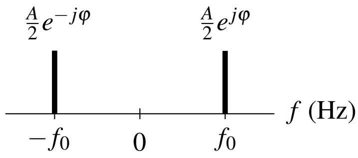

Figure 5. Line spectrum of the sinusoid $A\cos(2\pi f_0 t + \varphi)$.

If we denote the formula (4) by $X_T(f)$, it can be defined in MATLAB with the statement

```
XT = @(f)A*exp(1j*phi)/2*PT(f-f0)+A*exp(-1j*phi)/2*PT(f+f0);
```

Hence, it is easy to plot the magnitude spectrum $|X_T(f)|$ as shown in Figs. 6 and 7. By examining the graphs, we see that if the frequency f_0 is large,[c] there are two tallest peaks located at $f = \pm f_0$, corresponding to the two terms in Euler's formula for the cosine. We also see that the peaks do not interfere much with each other when f_0 is large, but when f_0 is small, it can be hard to tell if there are two peaks or just one large one.

[c] Here "large" is relative to the width of the central peak in $\mathrm{sinc}(Tf)$, whose width is $2/T$.

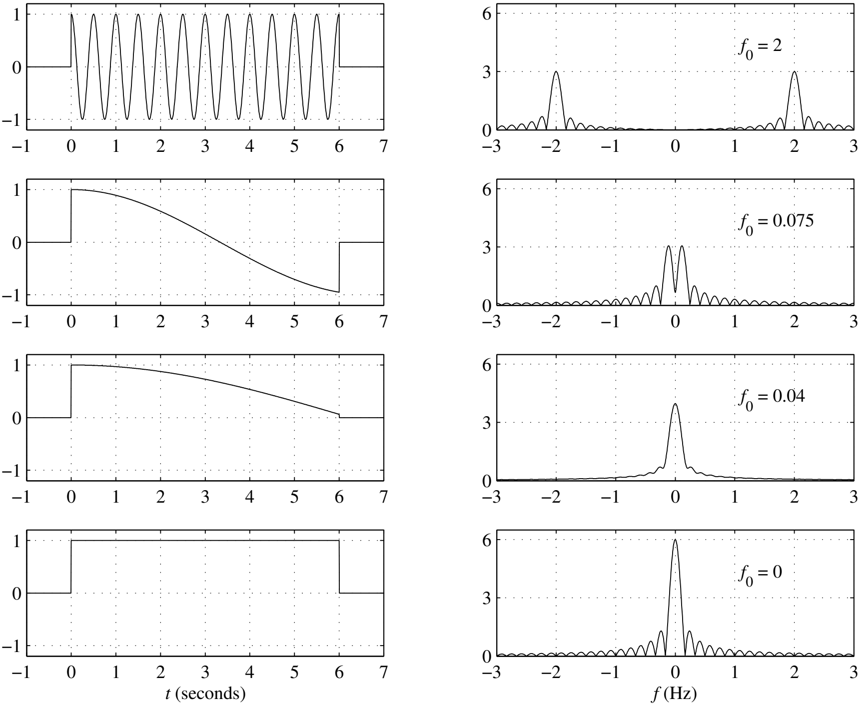

Figure 6. On the left, the waveform $A\cos(2\pi f_0 t + \varphi)$ is measured for $0 \leq t \leq T = 6$ and different values of the frequency f_0; the amplitude and phase are fixed at $A = 1$ and $\varphi = 0$, respectively. On the right, the corresponding magnitude spectra are shown.

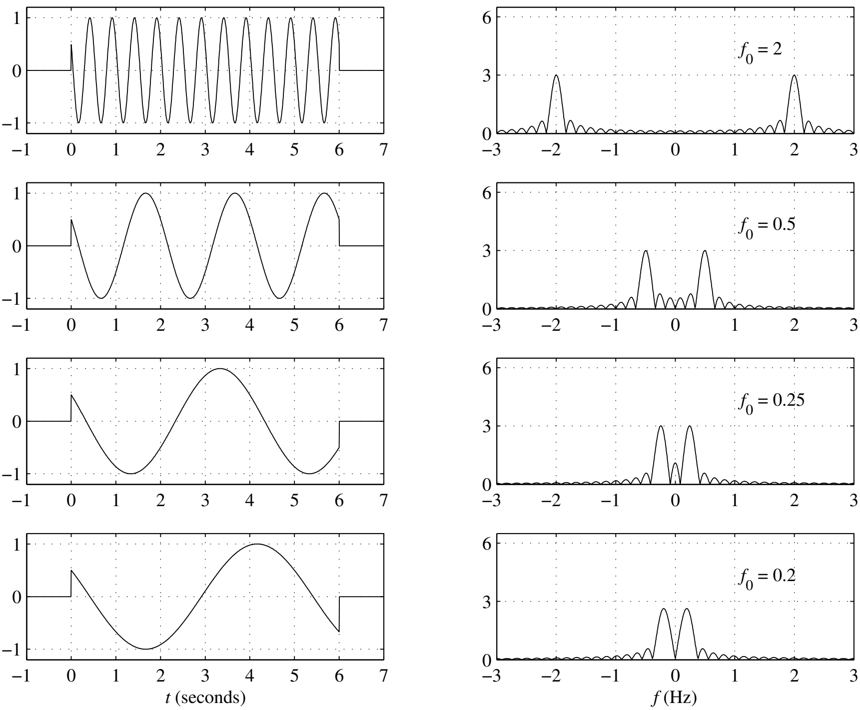

Figure 7. Same as Fig. 6 except that $\varphi = \pi/3$ and the frequencies are different.

4. The Fourier Series

We remarked in a footnote of Example 3 that there is no loss of generality in assuming that a signal defined on a finite interval, e.g., Fig. 8, has the form $\sum_k x_k e^{j2\pi f_k t}$.

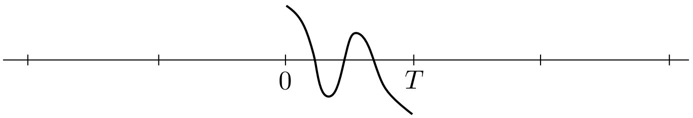

Figure 8. A finite-duration waveform $x(t)$ defined only for $0 \le t \le T$.

This is a consequence of the fact that essentially all such waveforms that arise in engineering practice have a **Fourier series** expansion

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{j2\pi(k/T)t}, \quad 0 \le t \le T, \tag{5}$$

where

$$x_k = \frac{1}{T} \int_0^T x(t) e^{-j2\pi(k/T)t}\, dt. \tag{6}$$

The formula (5) is called the **synthesis equation** or the **Fourier series expansion** of $x(t)$. The formula (6) is called the **analysis equation** or the **Fourier series coefficient formula**. Although $x(t)$ is only defined for $0 \le t \le T$, the series on the right in (5) makes sense for all $-\infty < t < \infty$ and is periodic with period T as shown in Fig. 9.

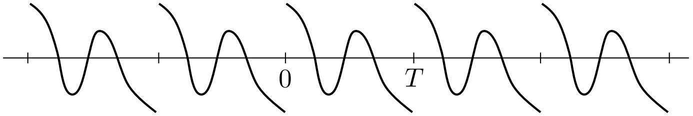

Figure 9. The Fourier series expansion is defined for $-\infty < t < \infty$, but it is equal to $x(t)$ in Fig. 8 only for $0 \le t \le T$.

Observe that the analysis equation (6) says that the kth Fourier series coefficient is equal to $(1/T)$ times the Fourier transform of $x(t)$ on $[0,T]$ evaluated at $f = k/T$. In other words, *to within the scale factor of* $1/T$, *the Fourier series coefficients are samples of the Fourier transform* (cf. italicized text at the end of Example 4).

Example 7. Consider the situations in Examples 4 and 5 in which three complex exponentials of frequencies f_a, f_b, and f_c are added together. The waveforms are observed over $0 \le t \le T = 6$, and their Fourier transforms are shown as the solid lines in Fig. 10. In addition, circles are placed on the transforms at frequencies that are multiples of $1/T$. In the first case, f_a, f_b, and f_c are all multiples of $1/T$. However, in the second case, the frequencies are not harmonically related; i.e., it is not the case that all of them are integer multiples of a common positive frequency.

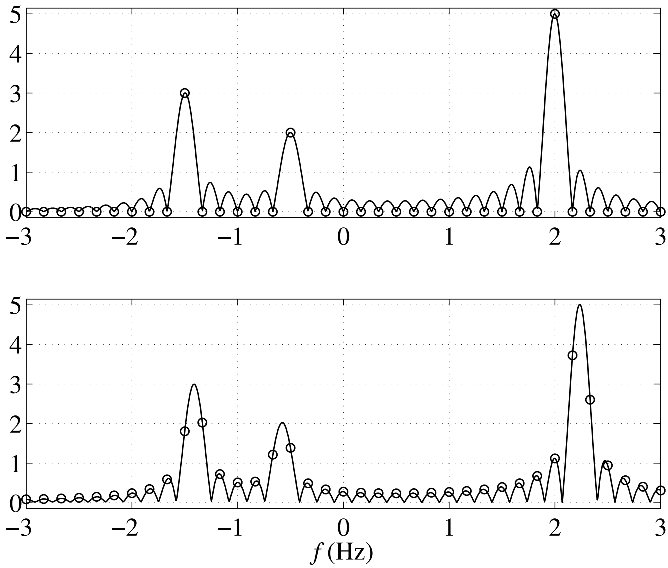

Figure 10. Fourier transforms (solid lines) and their samples (circles) at frequencies that are multiples of $1/T$ with $T = 6$ as described in Example 7.

Example 8 (Derivation of the Analysis Equation). *The reader should first review the properties of $P_T(f)$ in the paragraph preceding Example 2, and then review Example 3.* Applying Example 3 to the synthesis equation (5) shows that

$$\int_0^T x(t)e^{-j2\pi ft}\,dt = \sum_{k=-\infty}^{\infty} x_k P_T(f - k/T).$$

Now fix an integer m and put $f = m/T$ to get

$$\begin{aligned}\int_0^T x(t)e^{-j2\pi(m/T)t}\,dt &= \sum_{k=-\infty}^{\infty} x_k P_T(m/T - k/T)\\ &= \sum_{k=-\infty}^{\infty} x_k P_T((m-k)/T).\end{aligned}$$

In the above sum, when $k \neq m$, we are evaluating P_T at a nonzero integer multiple of $1/T$, which means that $P_T((m-k)/T) = 0$. Hence, there is only one nonzero term in the sum, and that is the term with $k = m$. Since $P_T(0) = T$,

$$\int_0^T x(t)e^{-j2\pi(m/T)t}\,dt = x_m \cdot T.$$

Dividing both sides by T yields the analysis equation (with m instead of k).

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5. Approximating the Spectrum Using Waveform Samples

If we sample a signal every Δt seconds, we call $f_s := 1/\Delta t$ the **sampling rate**. Recalling the definition of integral, we can approximate $X(f)$ by writing

$$\begin{aligned} X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt &\approx \sum_n x(n\Delta t)e^{-j2\pi f(n\Delta t)}\,\Delta t \\ &= \frac{1}{f_s}\sum_n x(n/f_s)e^{-j2\pi(f/f_s)n}. \end{aligned} \tag{7}$$

Notice that since both f and f_s have units of Hz, the quotient f/f_s has no units. If we put

$$\widetilde{X}(\widehat{f}\,) := \sum_n x(n/f_s)e^{-j2\pi\widehat{f}\,n}, \tag{8}$$

then the approximation in (7) becomes

$$X(f) \approx \frac{1}{f_s}\widetilde{X}(f/f_s). \tag{9}$$

The variable $\widehat{f}$ is called **digital frequency**, and it has no units. Furthermore, the formula for $\widetilde{X}(\widehat{f}\,)$ implies that it is a periodic function of $\widehat{f}$ with period one.[d] Hence, we only need to evaluate $\widetilde{X}(\widehat{f}\,)$ over an interval of length one. We usually take this interval to be $-1/2 \leq \widehat{f} \leq 1/2$. This means that in (9), $|f/f_s| \leq 1/2$, or $|f| \leq f_s/2$. In other words, the best we can hope for is to approximate $X(f)$ for $|f| \leq f_s/2$. In order for this to capture all the features of $X(f)$, we must have $X(f) = 0$ for $|f| > f_s/2$. In practice, we will often know or be able to arrange that $X(f) = 0$ for $|f|$ greater than

[d] Observe that

$$\widetilde{X}(\widehat{f}+1) = \sum_n x(n/f_s)e^{-j2\pi(\widehat{f}+1)n} = \sum_n x(n/f_s)e^{-j2\pi\widehat{f}\,n}\underbrace{e^{-j2\pi n}}_{=1} = \sum_n x(n/f_s)e^{-j2\pi\widehat{f}\,n} = \widetilde{X}(\widehat{f}\,).$$

some **cutoff frequency** f_{cutoff}.[e] If we then choose the sampling rate $f_s > 2f_{\text{cutoff}}$, it will be true that $X(f) = 0$ for $|f| > f_s/2$. The quantity $2f_{\text{cutoff}}$ is called the **Nyquist rate**, and the **Nyquist criterion** says that we must sample faster than the Nyquist rate; i.e., twice the cutoff frequency, in order to approximate $X(f)$.

Now suppose we are given only a finite number of samples, say $x(n/f_s)$ for $n = 0,\ldots,N-1$. Put $T := N/f_s$ and proceed as in (7) to get

$$\int_0^T x(t)e^{-j2\pi ft}\,dt \approx \frac{1}{f_s}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi f(n/f_s)}. \tag{10}$$

This suggests that we put

$$\widetilde{X}_N(\widehat{f}\,) := \sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi\widehat{f}\,n},$$

which has period one just like $\widetilde{X}(\widehat{f}\,)$ defined earlier. Hence, the right-hand side of (10) has period f_s. This is illustrated in Fig. 11, which plots the left- and right-hand sides of (10) for the waveform described in Example 4.

Using the definition of $\widetilde{X}_N(\widehat{f}\,)$, the approximation (10) becomes

$$\int_0^T x(t)e^{-j2\pi ft}\,dt \approx \frac{1}{f_s}\widetilde{X}_N(f/f_s).$$

We now take frequency samples by setting $f = kf_s/N$; i.e.,

$$\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=kf_s/N} \approx \frac{1}{f_s}\widetilde{X}_N(k/N) = \frac{1}{f_s}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi kn/N}. \tag{11}$$

The quantity

$$X[k] := \sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi kn/N} \tag{12}$$

is called the **discrete Fourier transform (DFT)**.[f] Putting this all together, we have the approximation

$$\boxed{\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=kf_s/N} \approx \frac{1}{f_s}X[k].} \tag{13}$$

[e] In this case, we say that the signal is **bandlimited** and that its **bandwidth** is f_{cutoff}.

[f] When we use the phrase "Fourier transform," we mean the integral (1) (or (2) when $x(t)$ is known only for $0 \le t \le T$). To refer to (12), we always write "DFT." Some authors refer to (1) as the **continuous-time Fourier transform (CTFT)**, and they refer to (8) as the **discrete-time Fourier transform (DTFT)**.

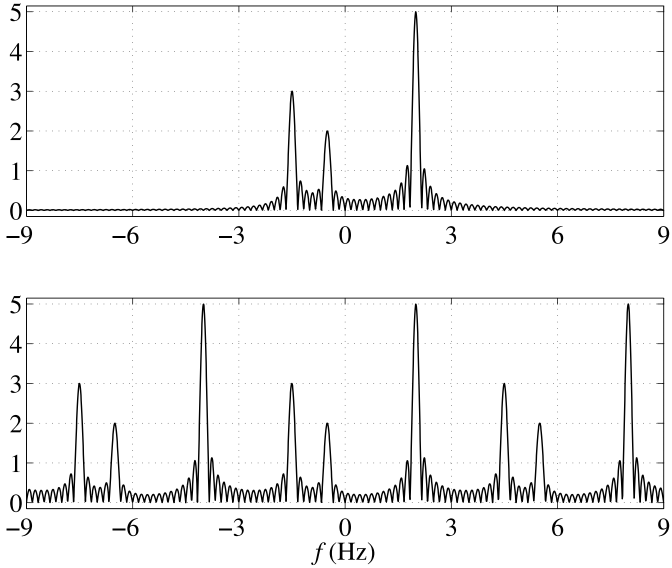

Figure 11. The top graph shows the magnitude of the left-hand side of (10), and the bottom graph shows the magnitude of the right-hand side of (10) for the waveform described in Example 4. To compute the right-hand side of (10), $f_s = 6$ and $N = 36$.

Note that because we put $T = N/f_s$, the frequencies $kf_s/N = k/T$ are the frequencies corresponding the the Fourier series coefficients x_k. This is illustrated in Fig. 12 for the situations in Examples 4 and 5. Notice that the top graph in Fig. 12 appears to be identical with the top graph in Fig. 10, while this is *not* true of the bottom graph. The equality of the top graphs is explained in Section 7.

Example 9. An FMRI waveform is sampled at rate $f_s = 4$ Hz, and $N = 122$ samples are recorded. The magnitude DFT values, $\left|X[k]\right|$, are plotted over the appropriate frequencies in the range $|f| \leq f_s/2 = 2$ as shown in Fig. 13. What conclusions would you draw?

5.1. Zero Padding

In some cases, the number of waveform samples N may be small, and this means that the frequency spacing of f_s/N in (13) will be large (e.g., Fig. 12). To get around

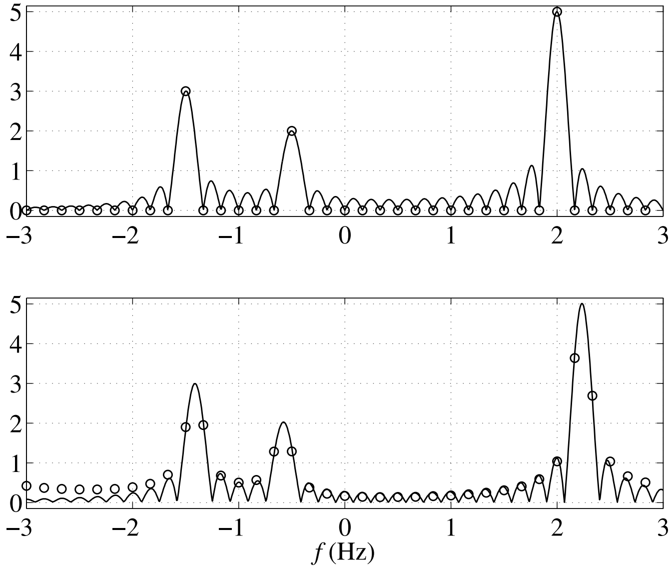

Figure 12. Magnitude spectra (solid lines) and DFT approximations (circles) from Examples 4 and 5. Here $f_s = 6$ and $N = 36$.

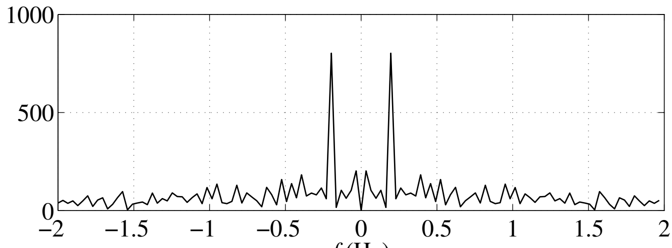

Figure 13. Magnitude DFT of FMRI data from Example 9 with $f_s = 4$ and $N = 122$.

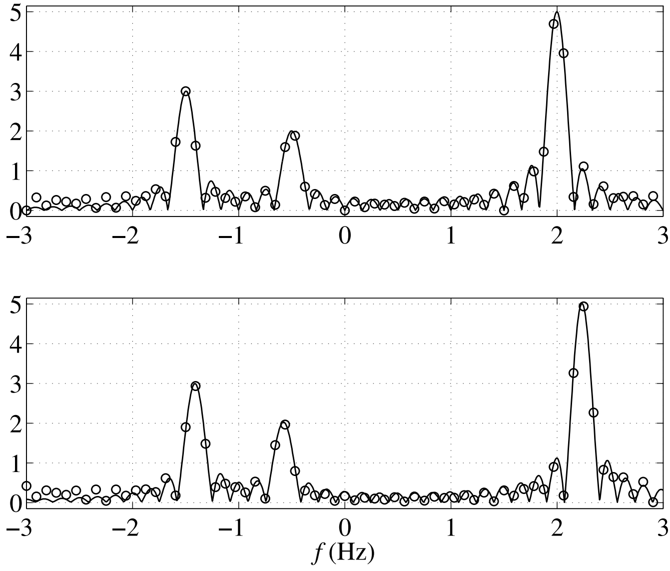

Figure 14. The top and bottom graphs show Fig. 12 redone with zero padding to $K = 64$.

this, we can choose a new length $K > N$ and put $x(n/f_s) := 0$ for $n = N, \dots, K-1$.[g] The DFT of this longer sequence is

$$\sum_{n=0}^{K-1} x(n/f_s)e^{-j2\pi kn/K} = \sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi kn/K}.$$

Comparing this with the right-hand side of (11), we see that

$$\left.\int_0^T x(t)e^{-j2\pi ft}\,dt\right|_{f=kf_s/K} \approx \frac{1}{f_s}\widetilde{X}_N(k/K) = \frac{1}{f_s}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi kn/K}.$$

In other words, zero padding provides a denser sampling of spectrum. For example, Fig. 12 is redone with $K = 64$ in Fig. 14 and with $K = 128$ in Fig. 15.

There are several reasons for zero padding:

- To reduce frequency spacing.
- To see the "ripples" in the continuous spectrum that are missing, for example, if you simply connect the circles in Fig. 12 with straight lines; compare Fig. 15.

[g] Remember, we only measured N samples of $x(t)$, we do not have access to the actual values of $x(t)$ any other values of t.

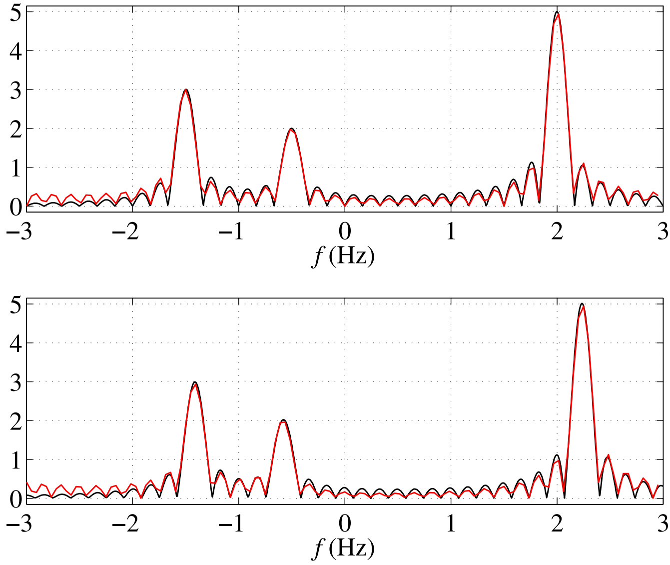

Figure 15. The top and bottom graphs show Fig. 12 redone with zero padding to $K = 128$.

- Since the FFT algorithm is most efficient when the length of the input is a power of two, it is common to zero pad to a power of two.
- Zero padding is essential for **fast convolution**, which we do not discuss here.

6. Some Messy Details about Using the DFT

Comparing (12) and (11) shows that $\widetilde{X}_N(k/N) = X[k]$. Since $\widetilde{X}_N(\widehat{f}\,)$ has period one, $X[k]$ has period N.[h] Hence, the restriction $-1/2 \le \widehat{f} \le 1/2$ corresponds to the restriction $-1/2 \le k/N \le 1/2$, or $-N/2 \le k \le N/2$. However, this range of k covers $N+1$ integers, and we only want one full period of integers, so we agree to use only $-N/2 \le k \le N/2-1$. The first challenge is that computer programs, such as `fft`[i] in MATLAB, return the vector of numbers[j]

$$\Big[X[0], \ldots, X[N/2-1], X[N/2], \ldots, X[N-1]\Big]$$

[h] To see this, write $X[k+N] = \widetilde{X}_N((k+N)/N) = \widetilde{X}_N(k/N+1) = \widetilde{X}_N(k/N) = X[k]$.
[i] The **fast Fourier transform (FFT)** is a clever algorithm for computing the DFT very quickly.
[j] We treat the case of even N here, but the MATLAB commands we give also handle the case of odd N.

rather than

$$\Big[X[-N/2],\ldots,X[-1],X[0],\ldots,X[N/2-1]\Big]. \tag{14}$$

However, since $X[k]$ has period N, the second half of the vector computed by MATLAB is equal to the first half of the vector we actually want. The MATLAB function `fftshift` will do the required rearrangement for us; i.e., if `x` is the vector of samples $x(n/f_s)$ for $n = 0,\ldots,N-1$, then the MATLAB statement

```
X = fftshift(fft(x));
```

puts the numbers in (14) into the vector `X`. Even when N is odd, MATLAB does the right thing. To plot `abs(X)/fs` with the appropriate Hertz frequencies on the horizontal axis, use

```
kvec0 = 0:N-1;
kvec = kvec0 - floor(N/2);
fhatvec = kvec/N;
fvec = fhatvec*fs;
plot(fvec,abs(X)/fs,'k')
```

To do zero padding to length `Nzp`, use

```
X = fftshift(fft(x,Nzp));
```

7. Understanding the Approximation (10)

To better understand the approximation in (10), we consider special cases in which both sides can be computed in closed form. To do this, we first need the following discrete analog of Example 1.

Example 10. Show that

$$\boxed{\widetilde{P}_N(\widehat{f}\,) := \sum_{n=0}^{N-1} e^{-j2\pi\widehat{f}\,n} = e^{-j\pi(N-1)\widehat{f}}\cdot N\cdot\frac{\operatorname{sinc}(N\widehat{f}\,)}{\operatorname{sinc}(\widehat{f}\,)},} \tag{15}$$

whenever $\widehat{f}$ is not an integer; for integers k, $|\widetilde{P}_N(k)| = N$. Furthermore, if k/N is not an integer, then $\widetilde{P}_N(k/N) = 0$.

Solution. For noninteger $\widehat{f}$, use the **geometric series** to write

$$\widetilde{P}(\widehat{f}\,) := \sum_{n=0}^{N-1} e^{-j2\pi\widehat{f}\,n} = \sum_{n=0}^{N-1}\big(e^{-j2\pi\widehat{f}}\big)^n = \frac{1-e^{-j2\pi\widehat{f}\,N}}{1-e^{-j2\pi\widehat{f}}}.$$

Proceeding as in Example1, we have

$$\widetilde{P}_N(\widehat{f}) = \frac{e^{-j\pi N\widehat{f}}}{e^{-j\pi\widehat{f}}} \cdot \frac{e^{j\pi N\widehat{f}} - e^{-j\pi N\widehat{f}}}{e^{j\pi\widehat{f}} - e^{-j\pi\widehat{f}}} = e^{-j\pi(N-1)\widehat{f}} \cdot \frac{\sin(\pi N\widehat{f})}{\sin(\pi\widehat{f})}$$
$$= e^{-j\pi(N-1)\widehat{f}} \cdot N \cdot \frac{\sin(\pi N\widehat{f})}{\sin(\pi\widehat{f})} \cdot \frac{\pi\widehat{f}}{\pi N\widehat{f}}$$
$$= e^{-j\pi(N-1)\widehat{f}} \cdot N \cdot \frac{\operatorname{sinc}(N\widehat{f})}{\operatorname{sinc}(\widehat{f})}.$$

The graph of $|\widetilde{P}_N(\widehat{f})| = N|\operatorname{sinc}(N\widehat{f})/\operatorname{sinc}(\widehat{f})|$ for $N = 10$ is shown in Fig. 16.

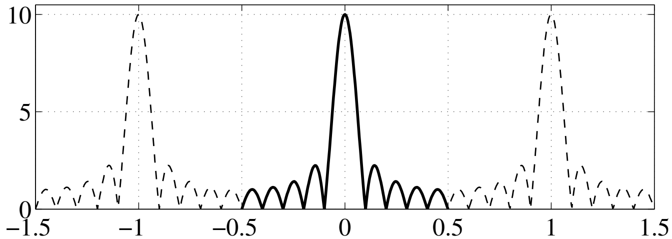

Figure 16. Graph of $|\widetilde{P}_N(\widehat{f})| = N|\operatorname{sinc}(N\widehat{f})/\operatorname{sinc}(\widehat{f})|$ for $N = 10$.

If we put $\widehat{f} = f/f_s$ in (15) and take $T := N/f_s$ as in the paragraph following (10), we obtain

$$\frac{1}{f_s}\widetilde{P}_N(f/f_s) = \frac{1}{f_s}e^{-j\pi(N-1)f/f_s} \cdot N \cdot \frac{\operatorname{sinc}(Nf/f_s)}{\operatorname{sinc}(f/f_s)}$$
$$= \frac{e^{j\pi f/f_s}}{\operatorname{sinc}(f/f_s)} \cdot e^{-j\pi Tf} \cdot T \cdot \operatorname{sinc}(Tf)$$
$$= \frac{e^{j\pi f/f_s}}{\operatorname{sinc}(f/f_s)} \cdot P_T(f). \tag{16}$$

Since we intend to keep $|f| \le f_s/2$ (equivalently, $|f/f_s| \le 1/2$), it is instructive to plot $\operatorname{sinc}(\widehat{f})$ for $|\widehat{f}| \le 1/2$ as shown in Fig. 17. We see that for $\widehat{f}$ in this range, $0.63 \le \operatorname{sinc}(\widehat{f}) \le 1$; and from the formula, the phase factor $e^{j\pi f/f_s}$ has an angle whose absolute value is at most $\pi/2$.

We now turn to the discrete analog of Example 3.

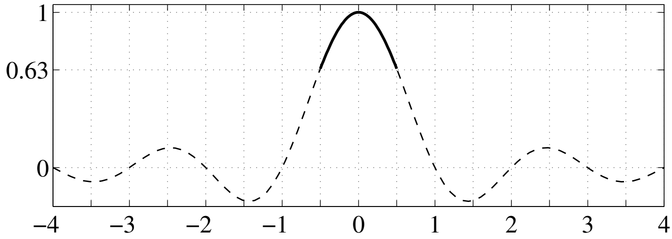

Figure 17. Graph of $\operatorname{sinc}(\widehat{f}\,)$.

Example 11. If

$$x(t) := \sum_k x_k e^{j2\pi f_k t}$$

is measured at sampling times n/f_s for $n = 0, \ldots, N-1$, show that the DTFT is

$$\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi \widehat{f}\, n} = \sum_k x_k \widetilde{P}_N(\widehat{f} - f_k/f_s). \tag{17}$$

Solution. Write

$$\begin{aligned}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi \widehat{f}\, n} &= \sum_{n=0}^{N-1}\left(\sum_k x_k e^{j2\pi f_k(n/f_s)}\right)e^{-j2\pi \widehat{f}\, n} \\ &= \sum_k x_k \left(\sum_{n=0}^{N-1} e^{-j2\pi(\widehat{f} - f_k/f_s)n}\right) \\ &= \sum_k x_k \widetilde{P}_N(\widehat{f} - f_k/f_s),\end{aligned}$$

where the last step follows from (15) with $\widehat{f}$ replaced by $\widehat{f} - f_k/f_s$.

We now substitute (17), with $\widehat{f}$ replaced by f/f_s, into the right-hand side of (10) to get

$$\begin{aligned}\int_0^T x(t)e^{-j2\pi f t}\,dt &\approx \frac{1}{f_s}\sum_k x_k \widetilde{P}_N((f-f_k)/f_s) \\ &= \sum_k x_k \frac{e^{j\pi(f-f_k)/f_s}}{\operatorname{sinc}((f-f_k)/f_s)} P_T(f-f_k), \quad \text{by (16)}, \end{aligned} \tag{18}$$

which contrasts nicely with the exact formula in Example 3. However, keep in mind that the right-hand side is periodic in f with period f_s (since $\widetilde{P}_N(\cdot)$ has period one). This was illustrated earlier in Fig. 11.

7.1. When the Approximation is Exact and the Sampling Theorem

We continue with $x(t)$ as in Example 11. Suppose that for some frequency f_0, each f_k has the form $f_k = kf_0$ and that $f_s = Nf_0$. Then $T := N/f_s = 1/f_0$. If for some i we take $f = if_0$, then (18) becomes

$$\begin{aligned}\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=if_0} &\approx \sum_k x_k \frac{e^{j\pi(i-k)/N}}{\text{sinc}((i-k)/N)} P_T((i-k)/T) \\ &= x_i \cdot P_T(0) = x_i \cdot T,\end{aligned}$$

where we recall that $P_T((i-k)/T) = 0$ for $k \neq i$ and $P_T(0) = T$. Furthermore, by the analysis equation (6), the left-hand side is also equal to $x_i T$. Hence, for $f = if_0$, the above approximation is actually exact; i.e.,

$$\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=if_0=i/T} = \frac{1}{f_s}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi\widehat{f}\,n}\bigg|_{\widehat{f}=if_0/f_s=i/N}. \tag{19}$$

Keep in mind that the above sum as a function of $\widehat{f}$ has period one and so we must have $|\widehat{f}\,| \leq 1/2$. Equivalently, we must have $|if_0/f_s| \leq 1/2$, or $|f_i| = |if_0| \leq f_s/2$. In other words, we need to have chosen $f_s = Nf_0 > 2\max\{|f_i|\}$. Putting this all together, we have the following result.

The Sampling Theorem for Periodic Signals. *Let*

$$x(t) := \sum_{m=-M}^{M} x_m e^{j2\pi m f_0 t}. \tag{20}$$

Choose any integer $N > 2M$ and put $f_s = Nf_0$.[k] *Then from knowledge of the samples $x(n/f_s)$ for $n = 0,\ldots,N-1$, we can recover $x(t)$ for all t.*

Proof. Since (19) holds for $i = -M,\ldots,M$, we can use the waveform samples $x(n/f_s)$ to compute the Fourier series coefficient integral on the left to obtain $x_{-M},\ldots,x_M$. Once these are know, we can compute $x(t)$ for any t using (20). □

Since $f_0/f_s = 1/N$, we can rewrite (19) as

$$\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=if_0=i/T} = \frac{1}{f_s}X[i].$$

In other words, the DFT exactly computes the Fourier series coefficients under the conditions of the sampling theorem.

[k] Note that $f_s = Nf_0 > 2Mf_0$, which is the Nyquist rate.

Remark. To see what happens when $x(t)$ in (20) is observed over L periods, write

$$x(t) = \sum_{m=-M}^{M} x_m e^{j2\pi(mL)(f_0/L)t}.$$

Put $\check{x}_k = x_m$ when $k = mL$ for some $m = -M, \ldots, M$, and put $\check{x}_k = 0$ otherwise. Then

$$x(t) = \sum_{k=-ML}^{ML} \check{x}_k e^{j2\pi k(f_0/L)t}.$$

Take $N > 2ML$ and put $f_s = N(f_0/L) > 2Mf_0$, which is the Nyquist rate. Then $T := N/f_s = L/f_0$, and (19) becomes

$$\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=i(f_0/L)=i/T} = \frac{1}{f_s}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi\widehat{f}\,n}\bigg|_{\widehat{f}=i(f_0/L)/f_s=i/N}.$$

Both sides are equal to $\check{x}_i$, which is zero unless $i = mL$ for some $m = -M, \ldots, M$.

References

[1] J. H. McClellan, R. W. Schafer, and M. A. Yoder, *DSP First*, 2nd ed. Boston: Pearson, 2016.

[2] Wikipedia, "Fourier series — Wikipedia, The Free Encyclopedia," [Online]. Available: https://en.wikipedia.org/w/index.php?title=Fourier_series&oldid=701781279https://en.wikipedia.org/w/index.php?title=Fourier_series&oldid=701781279, accessed Feb. 4, 2016.

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