Introduction to Frequency Analysis and the DFT (Short Version)

John A. Gubner
Department of Electrical and Computer Engineering
University of Wisconsin–Madison

Contents

1 Introduction **2**
2 The Fourier Transform **2**
3 The Sinc Function and the Rectangular Window **4**
4 The Effect of Rectangular Windowing **6**
5 Approximating the Spectrum Using Waveform Samples **8**
5.1 Zero Padding 10
6 Some Messy Details about Using the DFT **11**
References **12**
Index **13**

Abstract

The continuous-time Fourier transform is defined. It is shown that windowing has the effect of blurring the transform. Then sampling is introduced as a way to approximate the transform integral. The resulting approximation is seen to be periodic. Windowing and sampling are then combined to obtain the discrete Fourier transform (DFT). MATLAB code for plotting the fast Fourier transform (FFT) on the appropriate frequency axis is provided.

It is assumed that the reader is familiar with **Euler's formulas**

$$e^{j\theta} = \cos\theta + j\theta, \quad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \text{and} \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j},$$

and the continuous-time unit impulse (Dirac impulse) $\delta(t)$. The reader should also be familiar with convolution and the sifting property of the impulse, as well as the fact that the transform of a product is the convolution of the corresponding transforms.

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

1. Introduction

The goal of frequency analysis is to process a signal $x(t)$ and determine what frequencies are present. For example, the standard form of a **sinusoid** is

$$x(t) = A\cos(2\pi f_0 t + \varphi),$$

where $A > 0$ is the **amplitude**, f_0 is the **frequency**, measured in units of inverse seconds or Hertz (Hz), and φ is the **phase**, measured in radians. In this case, without knowing the formula for $x(t)$, and using only the values of $x(t)$, the goal of frequency analysis is to determine the values of A, f_0, and φ. Usually we only worry about A and f_0, since φ is only unique up to additive multiples of 2π. To add a bit more generality, suppose

$$x(t) = \sum_{k=1}^{K} A_k \cos(2\pi f_k t + \varphi_k).$$

We would like to use frequency analysis to tell us the number of terms K, the weights A_k and the corresponding frequencies f_k.

Our initial tool will be the **continuous-time Fourier transform (CTFT)** (or just FT when there is no ambiguity), which depends on the values of the waveform $x(t)$ for all times t.

Our first problem is that even if the waveform $x(t)$ is defined for all $-\infty < t < \infty$, we can only observe it for t in a finite time window of duration T, such as $[0,T]$ or $[-T/2, T/2]$. This leads us to study the effect of **windowing**. The main effect of windowing is to blur the original spectrum (sometimes known as **spectral leakage**).

The second problem we consider is that of approximating the transform integral using values of $x(t)$ only for discrete values of t, say $t = n\Delta t$ for integers n with $-\infty < n < \infty$ and some small time separation Δt. This leads us to study the effect of sampling, and introduces us to the **discrete-time Fourier transform (DTFT)**. The main effect of sampling is to introduce **periodicity** into the tranform approximation.

To conclude our discussion, we combine windowing and sampling, which leads us to the **discrete Fourier transform (DFT)**. This tool necessarily imposes both blurring (spectral leakage) and periodicity on the original CTFT.

2. The Fourier Transform

The **continuous-time Fourier transform (CTFT)**, or **spectrum**, of a waveform $x(t)$ is defined by

$$X(f) := \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t}\, dt,$$

and the inverse Fourier transform is given by

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t}\, df.$$

For example, the inverse transform of $X(f) = \delta(f - f_0)$ is easily computed to be

$$\int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t}\, df = e^{j2\pi f_0 t}.$$

We denote the fact that the time function $e^{j2\pi f_0 t}$ and the frequency function $\delta(f - f_0)$ are **transform pairs** by writing

$$\boxed{e^{j2\pi f_0 t} \overset{\mathsf{FT}}{\longleftrightarrow} \delta(f - f_0).} \tag{1}$$

This spectrum is graphed in Figure 1.

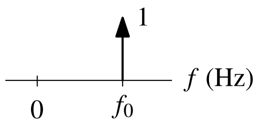

Figure 1. Spectrum of the complex exponential waveform $e^{j2\pi f_0 t}$.

By Euler's formula,

$$\cos(2\pi f_0 t) = \tfrac{1}{2} e^{j2\pi f_0 t} + \tfrac{1}{2} e^{j2\pi(-f_0)t}.$$

By linearity of the transform integral, it follows that

$$\boxed{\cos(2\pi f_0 t) \overset{\mathsf{FT}}{\longleftrightarrow} \tfrac{1}{2}\delta(f - f_0) + \tfrac{1}{2}\delta(f + f_0).}$$

This spectrum is graphed in Figure 2.

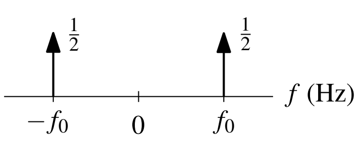

Figure 2. Spectrum of the sinusoid $\cos(2\pi f_0 t)$.

More generally, writing

$$A\cos(2\pi f_0 t + \varphi) = \tfrac{A}{2}e^{j\varphi}e^{j2\pi f_0 t} + \tfrac{A}{2}e^{-j\varphi}e^{j2\pi(-f_0)t},$$

it follows that

$$\boxed{A\cos(2\pi f_0 t + \varphi) \overset{\mathsf{FT}}{\longleftrightarrow} \tfrac{A}{2}e^{j\varphi}\delta(f - f_0) + \tfrac{A}{2}e^{-j\varphi}\delta(f + f_0).}$$

This spectrum is graphed in Figure 3.

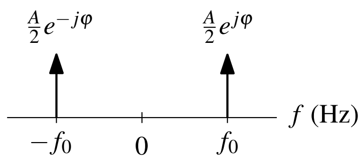

Figure 3. Spectrum of the sinusoid $A\cos(2\pi f_0 t + \varphi)$.

In practice, we never observe $x(t)$ for all $-\infty < t < \infty$. We typically have $x(t)$ only for t in a finite time interval of duration T, say $[0, T]$ or $[-T/2, T/2]$.

3. The Sinc Function and the Rectangular Window

The **sinc** function is essential to understand windowing. This function is defined by

$$\operatorname{sinc}(\theta) := \begin{cases} \dfrac{\sin(\pi\theta)}{\pi\theta}, & \theta \neq 0, \\ 1, & \theta = 0, \end{cases}$$

and is plotted in Figure 4. The reason for including the factor of π is to make the zero crossings occur when θ is a nonzero integer. Another important feature of $\operatorname{sinc}(\theta)$ is that the tallest peak, called the **main lobe**, occurs at $\theta = 0$, has height 1, and width 2.[a] The remaining peaks and valleys are called **sidelobes**.

Consider the symmetric **rectangular window**

$$r_T(t) := \begin{cases} 1, & |t| \leq T/2, \\ 0, & |t| > T/2, \end{cases}$$

which is graphed at the top in Figure 5.

[a] The width of a peak is measured between the zeros on either side. The tallest peak in Figure 4 sits between the zeros at $\theta = \pm 1$, so the width is $(+1) - (-1) = 2$.

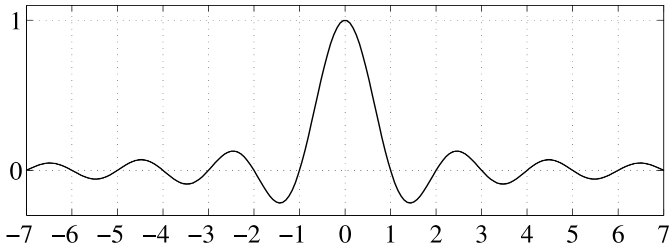

Figure 4. The function $\operatorname{sinc}(\theta)$.

Example 1. Verify the transform pair

$$\boxed{r_T(t) \stackrel{\mathsf{FT}}{\longleftrightarrow} R_T(f) := T\operatorname{sinc}(Tf),}$$

which is illustrated in Figure 5.

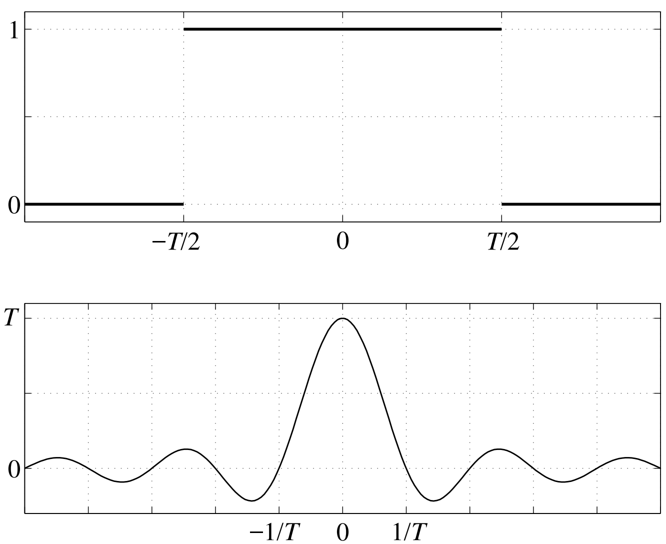

Figure 5. Symmetric rectangular window $r_T(t)$ (top) and its spectrum $T\operatorname{sinc}(Tf)$ (bottom).

Solution. We compute

$$\int_{-\infty}^{\infty} r_T(t)e^{-j2\pi ft}\,dt = \int_{-T/2}^{T/2} e^{-j2\pi ft}\,dt = \frac{e^{-j2\pi ft}}{-j2\pi f}\Bigg|_{t=-T/2}^{t=T/2} = \frac{e^{j\pi Tf}-e^{-j\pi Tf}}{2j\pi f}$$
$$= \frac{1}{\pi f}\cdot\frac{e^{j\pi Tf}-e^{-j\pi Tf}}{2j}$$
$$= T\frac{\sin(\pi Tf)}{\pi Tf}$$
$$= T\operatorname{sinc}(Tf).$$

4. The Effect of Rectangular Windowing

Consider a linear combination of complex exponentials, say

$$x(t) = ae^{j2\pi f_a t} + be^{j2\pi f_b t} + ce^{j2\pi f_c t}. \tag{2}$$

Taking the transform term by term, and using (1), we have

$$X(f) = a\delta(f-f_a) + b\delta(f-f_b) + c\delta(f-f_c),$$

which is illustrated in Figure 6. As long as the three frequencies f_a, f_b, and f_c are

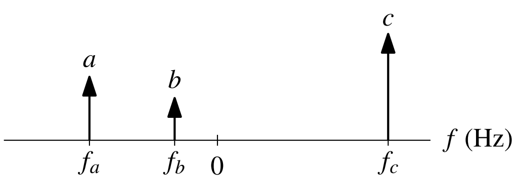

Figure 6. Spectrum of $x(t)$ in (2).

distinct, we can see distinct impulses, no matter how close the frequencies are to each other, and no matter what the relative strengths of a, b, and c are.

Now suppose we can only observe (2) for $t \in [-T/2, T/2]$ and compute

$$X_T(f) := \int_{-T/2}^{T/2} x(t)e^{-j2\pi ft}\,dt.$$

To understand what is going on, recall that $r_T(t) = 1$ for $-T/2 \le t \le T/2$ and $r_T(t) = 0$ otherwise. Hence, we can write

$$X_T(f) = \int_{-\infty}^{\infty} x(t)r_T(t)e^{-j2\pi ft}\,dt.$$

In other words, $X_T(f)$ is the transform of the product of the infinite-duration signal $x(t)$ and the finite-duration window function $r_T(t)$. We now use the transform property saying that the transform of a product is the convolution of the corresponding transforms,

$$\boxed{x(t)y(t) \xleftrightarrow{\mathsf{FT}} (X*Y)(f) := \int_{-\infty}^{\infty} X(\nu)Y(f-\nu)\,d\nu.}$$

Applying this result means we have to compute

$$\begin{aligned} X_T(f) &= \int_{-\infty}^{\infty} X(\nu)R_T(f-\nu)\,d\nu \\ &= \int_{-\infty}^{\infty} \big[a\delta(\nu-f_a)+b\delta(\nu-f_b)+c\delta(\nu-f_c)\big]R_T(f-\nu)\,d\nu. \end{aligned}$$

Now break up the integral into three separate integrals, and use the sifting property to evaluate each one. We find that

$$X_T(f) = aR_T(f-f_a)+bR_T(f-f_b)+cR_T(f-f_c),$$

which is illustrated in Figure 7. We infer from Figure 7 that $f_a = -1.5$ Hz, $f_b =$

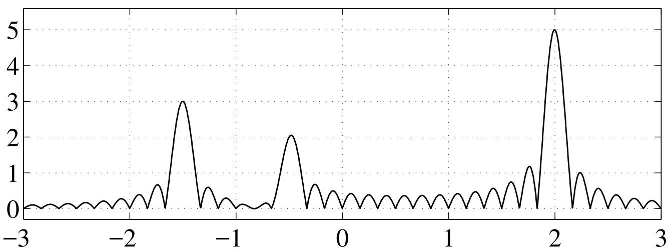

Figure 7. Absolute value of the spectrum of $x(t)r_T(t)$. Compare with the spectrum of $x(t)$ itself in Figure 6.

-0.5 Hz, and $f_c = 2$ Hz. Since the zero crossings of $R_T(f)$ occur at multiples of $1/T$, and we see that there are six crossing in $[0,1)$, we conclude that $T = 6$. Next, to determine the weights a, b, and c, we use the fact that the maximum value of $R_T(f)$ is T; hence, the three largest peaks in the graph have heights $aT = 3$, $bT = 2$, and $cT = 5$. Equivalently, $a = 3/6 = 1/2$, $b = 2/6 = 1/3$, and $c = 5/6$.

When we compare Figures 6 and 7, we see that each impulse is blurred or "leaks" into nearby frequencies. In other words, **windowing leads to *spectral leakage*.**

To give another example of the effect of windowing and spectral leakage, consider the ideal lowpass filter $H(f) := 1$ for $|f| \le f_c$ and $H(f) := 0$ otherwise. Taking the inverse Fourier transform (following the method of Example 1) shows that $h(t) = 2f_c \operatorname{sinc}(2f_c t)$. In Figure 8, we compare the transform $H(f)$ (dashed line) with the windowed approximation

$$\int_{-T/2}^{T/2} h(t)e^{-j2\pi ft}\, dt \tag{3}$$

(solid line), which is computed using numerical integration.

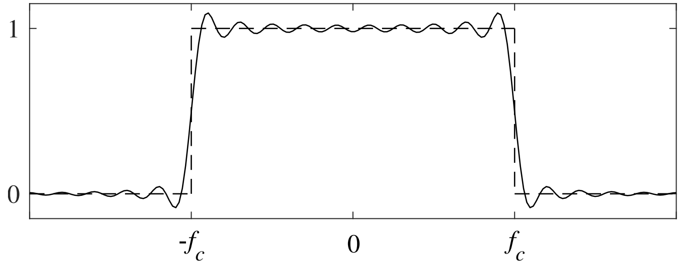

Figure 8. Ideal lowpass filter (dashed line) and its approximation using the windowed Fourier transform (3) (solid line).

5. Approximating the Spectrum Using Waveform Samples

If we sample a signal every Δt seconds, we call $f_s := 1/\Delta t$ the **sampling rate**. Recalling the definition of integral, we can approximate $X(f)$ by writing

$$\begin{aligned} X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\, dt &\approx \sum_n x(n\Delta t)e^{-j2\pi f(n\Delta t)}\,\Delta t \\ &= \frac{1}{f_s}\sum_n x(n/f_s)e^{-j2\pi(f/f_s)n}. \end{aligned} \tag{4}$$

Notice that since both f and f_s have units of Hz, the quotient f/f_s has no units. If we put

$$\widetilde{X}(\widehat{f}) := \sum_n x(n/f_s)e^{-j2\pi\widehat{f}\,n}, \tag{5}$$

then the approximation in (4) becomes

$$X(f) \approx \frac{1}{f_s}\widetilde{X}(f/f_s). \tag{6}$$

The variable $\widehat{f}$ is called **digital frequency**, and it has no units. Furthermore, the formula for $\widetilde{X}(\widehat{f})$ implies that it is a periodic function of $\widehat{f}$ with period one.[b] Hence, we only need to evaluate $\widetilde{X}(\widehat{f})$ over a frequency interval of length one. We usually take this interval to be $-1/2 \leq \widehat{f} \leq 1/2$. This means that in (6), $|f/f_s| \leq 1/2$, or $|f| \leq f_s/2$. In other words, the best we can hope for is to approximate $X(f)$ for $|f| \leq f_s/2$. In order for this to capture all the features of $X(f)$, we must have $X(f) = 0$ for $|f| > f_s/2$. In practice, we will often know or be able to arrange that $X(f) = 0$ for $|f|$ greater than some **cutoff frequency** f_{cutoff}.[c] If we then choose the sampling rate $f_s > 2f_{\text{cutoff}}$, it will be true that $X(f) = 0$ for $|f| > f_s/2$. The quantity $2f_{\text{cutoff}}$ is called the **Nyquist rate**, and the **Nyquist criterion** says that we must sample faster than the Nyquist rate; i.e., twice the cutoff frequency, in order to approximate $X(f)$.

Now suppose we are given only a finite number of samples, say $x(n/f_s)$ for $n = 0,\ldots,N-1$. Put $T := N/f_s$ and proceed as in (4) to get

$$\int_0^T x(t)e^{-j2\pi ft}\,dt \approx \frac{1}{f_s}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi f(n/f_s)}. \tag{7}$$

This suggests that we put

$$\widetilde{X}_N(\widehat{f}) := \sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi\widehat{f}\,n},$$

which has period one just like $\widetilde{X}(\widehat{f})$ defined earlier. Hence, the right-hand side of (7) has period f_s. This is illustrated in Figure 9, which plots the left- and right-hand sides of (7) for the waveform discussed in Section 4.

Using the definition of $\widetilde{X}_N(\widehat{f})$, the approximation (7) becomes

$$\int_0^T x(t)e^{-j2\pi ft}\,dt \approx \frac{1}{f_s}\widetilde{X}_N(f/f_s).$$

We now take frequency samples by setting $f = kf_s/N$; i.e.,

$$\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=kf_s/N} \approx \frac{1}{f_s}\widetilde{X}_N(k/N) = \frac{1}{f_s}\sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi kn/N}. \tag{8}$$

---

[b] Observe that

$$\widetilde{X}(\widehat{f}+1) = \sum_n x(n/f_s)e^{-j2\pi(\widehat{f}+1)n} = \sum_n x(n/f_s)e^{-j2\pi\widehat{f}\,n}\underbrace{e^{-j2\pi n}}_{=1} = \sum_n x(n/f_s)e^{-j2\pi\widehat{f}\,n} = \widetilde{X}(\widehat{f}).$$

[c] In this case, we say that the signal is **bandlimited** and that its **bandwidth** is f_{cutoff}.

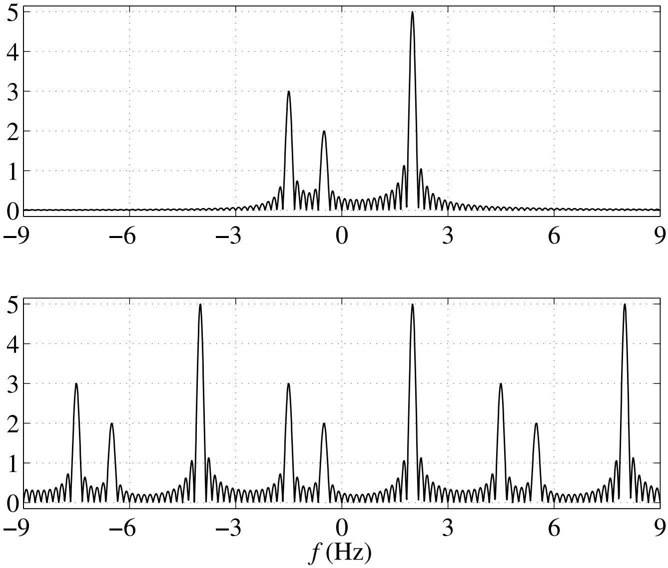

Figure 9. The top graph shows the magnitude of the left-hand side of (7), and the bottom graph shows the magnitude of the right-hand side of (7) for the waveform discussed in Section 4. To compute the right-hand side of (7), $f_s = 6$ and $N = 36$.

The quantity

$$X[k] := \sum_{n=0}^{N-1} x(n/f_s)e^{-j2\pi kn/N} \tag{9}$$

is called the **discrete Fourier transform (DFT)**. Putting this all together, we have the approximation

$$\boxed{\int_0^T x(t)e^{-j2\pi ft}\,dt\bigg|_{f=kf_s/N} \approx \frac{1}{f_s}X[k].} \tag{10}$$

Example 2. An FMRI waveform is sampled at rate $f_s = 4$ Hz, and $N = 122$ samples are recorded. The magnitude DFT values, $\left|X[k]\right|$, are plotted over the appropriate frequencies in the range $|f| \leq f_s/2 = 2$ as shown in Figure 10. What conclusions would you draw?

5.1. Zero Padding

In some cases, the number of waveform samples N may be small, and this means that the frequency spacing of f_s/N in (10) will be large. To get around this, we can

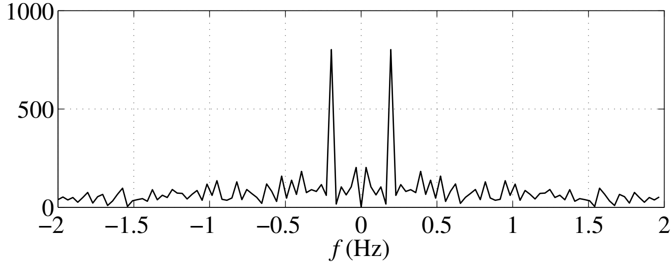

Figure 10. Magnitude DFT of FMRI data from Example 2 with $f_s = 4$ and $N = 122$.

choose a new length $K > N$ and put $x(n/f_s) := 0$ for $n = N, \ldots, K-1$.[d] The DFT of this longer sequence is

$$\sum_{n=0}^{K-1} x(n/f_s) e^{-j2\pi kn/K} = \sum_{n=0}^{N-1} x(n/f_s) e^{-j2\pi kn/K}.$$

Comparing this with the right-hand side of (8), we see that

$$\int_0^T x(t) e^{-j2\pi ft}\, dt \bigg|_{f=kf_s/K} \approx \frac{1}{f_s} \widetilde{X}_N(k/K) = \frac{1}{f_s} \sum_{n=0}^{N-1} x(n/f_s) e^{-j2\pi kn/K}.$$

In other words, zero padding provides a denser sampling of spectrum.

There are several reasons for zero padding:

- To reduce frequency spacing.
- Since the FFT algorithm is most efficient when the length of the input is a power of two, it is common to zero pad to a power of two.
- Zero padding is essential for **fast convolution**, which we do not discuss here.

6. Some Messy Details about Using the DFT

Comparing (9) and (8) shows that $\widetilde{X}_N(k/N) = X[k]$. Since $\widetilde{X}_N(\widehat{f}\,)$ has period one, $X[k]$ has period N.[e] Hence, the restriction $-1/2 \leq \widehat{f} \leq 1/2$ corresponds to the restriction $-1/2 \leq k/N \leq 1/2$, or $-N/2 \leq k \leq N/2$. However, this range of k covers $N+1$ integers, and we only want one full period of integers, so we agree to use only

[d] Remember, we only measured N samples of $x(t)$, we do not have access to the actual values of $x(t)$ any other values of t.

[e] To see this, write $X[k+N] = \widetilde{X}_N((k+N)/N) = \widetilde{X}_N(k/N+1) = \widetilde{X}_N(k/N) = X[k]$.

$-N/2 \le k \le N/2-1$. The first challenge is that computer programs, such as `fft`[f] in MATLAB, return the vector of numbers[g]

$$\Big[X[0],\ldots,X[N/2-1],X[N/2],\ldots,X[N-1]\Big]$$

rather than

$$\Big[X[-N/2],\ldots,X[-1],X[0],\ldots,X[N/2-1]\Big]. \tag{11}$$

However, since $X[k]$ has period N, the second half of the vector computed by MATLAB is equal to the first half of the vector we actually want. The MATLAB function `fftshift` will do the required rearrangement for us; i.e., if `x` is the vector of samples $x(n/f_s)$ for $n = 0,\ldots,N-1$, then the MATLAB statement

```
X = fftshift(fft(x));
```

puts the numbers in (11) into the vector `X`. Even when N is odd, MATLAB does the right thing. To plot `abs(X)/fs` with the appropriate Hertz frequencies on the horizontal axis, use

```
kvec0 = 0:N-1;
kvec = kvec0 - floor(N/2);
fhatvec = kvec/N;
fvec = fhatvec*fs;
plot(fvec,abs(X)/fs,'k')
```

To do zero padding to length `Nzp`, use

```
X = fftshift(fft(x,Nzp));
```

References

[1] J. H. McClellan, R. W. Schafer, and M. A. Yoder, *DSP First*, 2nd ed. Boston: Pearson, 2016.

[2] A. H. Nuttall, "Some windows with very good sidelobe behavior," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 29, no. 1, pp. 84–91, Feb. 1981.

[f] The **fast Fourier transform (FFT)** is a clever algorithm for computing the DFT very quickly.

[g] We treat the case of even N here, but the MATLAB commands we give also handle the case of odd N.

Index

amplitude, 2

bandlimited waveform, 9
bandwidth, 9

continuous-time Fourier transform (CTFT), 2
cutoff frequency, 9

DFT, *see* discrete Fourier transform
digital frequency, 9
discrete Fourier transform (DFT), 2, 10
discrete-time Fourier transform (DTFT), 2

Euler's formulas, 1

fast convolution, 11
fast Fourier transform (FFT), 12
FFT, *see* fast Fourier transform
frequency, 2

main lobe, 4

Nyquist
  criterion, 9
  rate, 9

periodicity, 2
phase, 2

rectangular window, 4

sampling rate, 8
sidelobes, 4
sinc function, 4
sinusoid, 2
  amplitude, 2
  frequency, 2
  phase, 2
  standard form, 2
spectral leakage, 2, 7
spectrum, 2

transform pairs, 3

windowing, 2