The Fundamental Theorem of Algebra and the Minimum Modulus Principle

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Abstract

A direct proof of the fundamental theorem of algebra is given. In other words, we show that every polynomial of degree greater than or equal to one has at least one root in the complex plane. The reader is assumed to be familiar with the following facts.

- The triangle inequality [2, pp. 14–15, Theorem 1.13 and p. 23, Problem 13]: For complex a and b,

$$\big||a|-|b|\big| \leq |a+b| \leq |a|+|b|.$$

- The binomial theorem: For complex a and b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad n = 0, 1, \ldots.$$

- Closed and bounded subsets of the plane are compact (the Heine–Borel Theorem) [2, p. 40, Theorem 2.41].
- A continuous function on a compact set is bounded and achieves its minimum and maximum values on the set [2, pp. 89–90, Theorem 4.16].

A slight modification of the proof yields the minimum modulus principle.

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

1. Preliminary Lemmas

Lemma 1. *If p is a polynomial of degree $n \geq 1$, then $|p(z)| \to \infty$ as $|z| \to \infty$.*

Proof. Suppose $p(z) = \sum_{k=0}^{n} c_k z^k$, where $c_n \neq 0$. Following [1, p. 140], write

$$p(z) = z^n \left[c_n + \frac{c_{n-1}}{z} + \cdots + \frac{c_0}{z^n} \right].$$

The key idea is that for large $|z|$, the terms with z in the denominator are negligible compared with c_n; i.e., $p(z) \approx z^n[c_n + 0]$, and so $|p(z)| \approx |c_n|\,|z|^n \to \infty$ as $|z| \to \infty$. To make this precise, let $|z|$ be so large that $\left| c_{n-1}/z + \cdots + c_0/z^n \right| < |c_n|/2$. Then

$$\begin{aligned}
|p(z)| &= |z|^n \left| c_n + \left(\frac{c_{n-1}}{z} + \cdots + \frac{c_0}{z^n} \right) \right| \\
&\geq |z|^n \left[|c_n| - \left| \frac{c_{n-1}}{z} + \cdots + \frac{c_0}{z^n} \right| \right], \quad \text{by the triangle inequality,} \\
&\geq |z|^n \big[|c_n| - |c_n|/2 \big] = |z|^n |c_n|/2 \to \infty.
\end{aligned}$$

□

Lemma 2 (Taylor's Theorem for Polynomials). *If p is a polynomial of degree $n \geq 1$, then for any $z_0 \in \mathbb{C}$,*

$$p(z) = p(z_0) + \sum_{\ell=1}^{n} \frac{p^{(\ell)}(z_0)}{\ell!} (z - z_0)^\ell.$$

Proof. Suppose $p(z) = \sum_{k=0}^{n} c_k z^k$. Then

$$p^{(\ell)}(z) = \sum_{k=\ell}^{n} k(k-1)\cdots(k-[\ell-1]) c_k z^{k-\ell} = \sum_{k=\ell}^{n} \frac{k!}{(k-\ell)!} c_k z^{k-\ell}.$$

Now write

$$\begin{aligned}
\sum_{\ell=0}^{n} \frac{p^{(\ell)}(z_0)}{\ell!} (z - z_0)^\ell &= \sum_{\ell=0}^{n} \frac{1}{\ell!} (z - z_0)^\ell \sum_{k=\ell}^{n} \frac{k!}{(k-\ell)!} c_k z_0^{k-\ell} \\
&= \sum_{k=0}^{n} c_k \sum_{\ell=0}^{k} \binom{k}{\ell} (z - z_0)^\ell z_0^{k-\ell} \\
&= \sum_{k=0}^{n} c_k [(z - z_0) + z_0]^k, \quad \text{by the binomial theorem,} \\
&= \sum_{k=0}^{n} c_k z^k = p(z).
\end{aligned}$$

□

Lemma 3. *Suppose*

$$f(z) = ae^{j\alpha} + be^{j\beta}(z - z_0)^\ell + (z - z_0)^{\ell+1}\varphi(z),$$

where a and b are positive, and $\ell \geq 1$. If φ is continuous on some closed disk D_0 centered at z_0, then there is some $z \in D_0$ with $|f(z)| < |f(z_0)|$.

Proof. (Based on [3].) Since φ is continuous on the compact set D_0, φ is bounded there, say by M, and we may write $|\varphi(z)| \leq M$ for all $z \in D_0$. Consider z of the form $z = z_0 + re^{j\theta}$, where r is small enough that $z \in D_0$. Then use the triangle inequality to write

$$\begin{aligned}
|f(z)| &\leq \left|ae^{j\alpha} + be^{j\beta}(z - z_0)^\ell\right| + |z - z_0|^{\ell+1}M \\
&= \left|ae^{j\alpha} + be^{j\beta}r^\ell e^{j\theta\ell}\right| + r^{\ell+1}M \\
&= \left|e^{j\alpha}\left(a + br^\ell e^{j(\beta+\theta\ell-\alpha)}\right)\right| + r^{\ell+1}M \\
&= \left|a + br^\ell e^{j(\beta+\theta\ell-\alpha)}\right| + r^{\ell+1}M.
\end{aligned}$$

Choose θ so that $e^{j(\beta+\theta\ell-\alpha)} = -1$; i.e., $\theta := (\pi + \alpha - \beta)/\ell$. Then

$$|f(z)| \leq |a - br^\ell| + r^{\ell+1}M.$$

Now further reduce r so that $br^\ell < a$ and $rM < b$. We now have

$$|f(z)| \leq a - br^\ell + r^{\ell+1}M = a - r^\ell[b - rM] < a = |f(z_0)|. \qquad \square$$

2. The Fundamental Theorem of Algebra

Theorem 4 (Fundamental Theorem of Algebra). *If p is a polynomial of degree greater than or equal to one, then p has a root in the complex plane.*

Proof. (Based on [3].) By Lemma 1, we can choose R so large that for all $|z| \geq R$, we have $|p(z)| > |p(0)|$. Since $D := \{z \in \mathbb{C} : |z| \leq R\}$ is closed and bounded, and since $|p(z)|$ is continuous, $|p(z)|$ achieves its minimum value on D at some $z_0 \in D$. We claim that $p(z_0) = 0$; i.e., z_0 is the required root. To obtain a contradiction, suppose otherwise that $p(z_0) \neq 0$.

Since $z_0 \in D$, $|z_0| \leq R$. In fact, $|z_0| < R$. To see this, notice that points on the boundary of D, i.e., points with $|z| = R$, have $|p(z)| > |p(0)|$ and so cannot minimize $|p(z)|$ on D.

With minimizer z_0, Taylor's Theorem for Polynomials implies

$$p(z) = ae^{j\alpha} + \sum_{k=1}^{n} c_k(z - z_0)^k,$$

where $c_k := p^{(k)}(z_0)/k!$, and writing $p(z_0) = ae^{j\alpha}$ for some $a > 0$ is justified because we assumed $p(z_0) \neq 0$. Next, we know that $c_n \neq 0$ since $\deg p = n \geq 1$. However, some of the other coefficients may be zero. Suppose $c_k = 0$ for $k = 1, \ldots, \ell - 1$, but $c_\ell \neq 0$. Then writing $c_\ell = be^{j\beta}$, we have

$$p(z) = ae^{j\alpha} + be^{j\beta}(z - z_0)^\ell + (z - z_0)^{\ell+1} \underbrace{\sum_{k=\ell+1}^{n} c_k (z - z_0)^{k-(\ell+1)}}_{=: \varphi(z)}.$$

Since $|z_0| < R$, there is a closed disk D_0 centered at z_0 and satisfying $D_0 \subset D$. Since φ is continuous on D_0, by Lemma 3, there is some $z \in D_0 \subset D$ with $|p(z)| < |p(z_0)|$, which contradicts z_0 minimizing p on D. □

3. The Minimum Modulus Principle

Theorem 5. *Let f be a continuous function on a closed disk D, and suppose f is analytic on the interior of D. If f is nonzero on D, then the minimum of $|f|$ on D is achieved on the boundary of D.*

Proof. Since $|f|$ is continuous on the closed and bounded set D, the minimum of $|f|$ on D is achieved by some $z_0 \in D$. We must show that z_0 lies on the boundary of D. Suppose otherwise that z_0 is an interior point of D. Since f is analytic on the interior of D, we can expand f in a Taylor series about z_0,

$$f(z) = f(z_0) + \sum_{k=1}^{\infty} c_k (z - z_0)^k,$$

where $c_k := f^{(k)}(z_0)/k!$. If all the $c_k = 0$, then f is constant on D, and the theorem is trivially true. Otherwise, let $\ell := \min\{k \geq 1 : c_k \neq 0\}$. Writing $c_\ell = be^{j\beta}$ and $f(z_0) = ae^{j\alpha}$, we have

$$f(z) = ae^{j\alpha} + be^{j\beta}(z - z_0)^\ell + (z - z_0)^{\ell+1} \underbrace{\sum_{k=\ell+1}^{\infty} c_k (z - z_0)^{k-(\ell+1)}}_{=: \varphi(z)}.$$

This expansion is only valid in a neighborhood of z_0, but we may restrict z to a closed disk D_0 centered at z_0 such that D_0 is a subset of the interior of D. Then φ is continuous on $D_0 \subset D$, and by Lemma 3, there is a $z \in D_0 \subset D$ with $|f(z)| < a = |f(z_0)|$. This contradicts the fact that z_0 minimizes f on D. □

References

[1] R. V. Churchill, J. W. Brown, and R. F. Verhey, *Complex Variables and Applications*, 3rd ed. New York: McGraw-Hill, 1976.

[2] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill, 1976.

[3] Wikipedia, "Fundamental theorem of algebra — Wikipedia, The Free Encyclopedia," [Online]. Available: `http://en.wikipedia.org/w/index.php?title=Fundamental_theorem_of_algebra&oldid=624510218`, accessed Oct. 1, 2014.