The Intermediate-Value Theorem

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Abstract

A simple proof of the intermediate-value theorem is given. As an easy corollary, we establish the existence of nth roots of positive numbers. It is assumed that the reader is familiar with the following facts and concepts from analysis:

- Let A be a nonempty set of real numbers bounded above. Then A has a least upper bound x_0 [1].
- A function f is continuous at x_0 if given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all x with $|x - x_0| < \delta$, $|f(x) - f(x_0)| < \varepsilon$. This last inequality can be rewritten as

$$-\varepsilon < f(x) - f(x_0) < \varepsilon$$

or, by multiplying through by -1,

$$-\varepsilon < f(x_0) - f(x) < \varepsilon,$$

from which we get

$$-\varepsilon + f(x) < f(x_0) < f(x) + \varepsilon.$$

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

References

[1] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill, 1976.

The Intermediate-Value Theorem. *Let f be a real-valued, continuous function defined on a finite interval $[a, b]$. Then f takes all values between $f(a)$ and $f(b)$.*

Proof. Without loss of generality, suppose $f(a) < y_0 < f(b)$. Put $A := \{x \in [a, b] : f(x) \leq y_0\}$. Since $f(a) < y_0$, $a \in A$ and we see that A is nonempty. By definition, A is bounded above. Therefore, the least upper bound axiom tells us that A has a least upper bound, which we denote by x_0. We show below that $f(x_0) = y_0$.

We start by showing that $f(x_0)$ is defined; i.e., we show that $x_0 \in [a, b]$. Since b is an upper bound of A, the least upper bound $x_0 \leq b$. Since x_0 is an upper bound of A and since $a \in A$, we must have $a \leq x_0$.

Let $\varepsilon > 0$ be given. Continuity of f at x_0 implies that there is an $\delta > 0$ such that for all $x \in [a, b]$ with $|x - x_0| < \delta$, we have $|f(x) - f(x_0)| < \varepsilon$. Equivalently,

$$-\varepsilon + f(x) < f(x_0) < f(x) + \varepsilon. \tag{1}$$

Since x_0 is the *least* upper bound of A, $x_0 - \delta$ is not an upper bound of A; hence, there is some $x_1 \in A$ with $x_0 - \delta < x_1 \leq x_0$. Since $|x_1 - x_0| < \delta$, we have from the right-hand inequality in (1) that

$$f(x_0) < f(x_1) + \varepsilon \leq y_0 + \varepsilon, \quad \text{since } x_1 \in A.$$

Since ε was arbitrary, it must be that $f(x_0) \leq y_0$. This further implies that $x_0 < b$; otherwise, if $x_0 = b$, we would have $y_0 < f(b) = f(x_0) \leq y_0$. Now that we have $x_0 < b$, choose any x_2 with $x_0 < x_2 < b$ and $|x_2 - x_0| < \delta$. Then by the left-hand inequality in (1),

$$f(x_0) > f(x_2) - \varepsilon > y_0 - \varepsilon, \quad \text{since } x_2 > x_0 \text{ implies } x_2 \notin A.$$

Since ε was arbitrary, it must be that $f(x_0) \geq y_0$. □

Lemma 1. *If $0 < a < b < \infty$, then $a^n < b^n$ for $n = 1, 2, \ldots$.*

Proof. Put $t := a/b < 1$. Repeatedly multiplying this inequality by t yields $t^2 < t$, $t^3 < t^2$, etc. Hence, $t^n < \cdots < t^2 < 1$. It follows that $a^n/b^n < 1$ or $a^n < b^n$. □

Corollary 2 (nth Roots). *Every positive real number has a unique positive nth root for $n = 1, 2, \ldots$.*

Proof. Given $y_0 > 0$, we must prove the existence of $x_0 > 0$ with $x_0^n = y_0$. If $y_0 = 1$, we can take $x_0 = 1$. If $y_0 < 1$, then $a := y_0/2 < y_0 < 1$ satisfies $a^n < a < y_0$. With $b := 1$, we have $a^n < y_0 < 1 = b^n$. Now apply the intermediate-value theorem to the continuous function $f(x) := x^n$ on $[a, b]$ to get the existence of x_0. By the lemma, x_0 is unique. For $y_0 > 1$, let u solve $f(u) = 1/y_0$; i.e., $u^n = 1/y_0$ or $(1/u)^n = y_0$. □