Magnitude and Phase of Complex Numbers

John A. Gubner
Department of Electrical and Computer Engineering
University of Wisconsin–Madison

Abstract

Every nonzero complex number can be expressed in terms of its magnitude and angle. This angle is sometimes called the phase or argument of the complex number. Although formulas for the angle of a complex number are a bit complicated, the angle has some properties that are simple to describe. In particular, when the complex number is a function of frequency, we derive a simple formula for the derivative of the argument.

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

Properties of the Angle of a Complex Number

Recall that every nonzero complex number $z = x + jy$ can be written in the form $re^{j\theta}$, where $r := |z| := \sqrt{x^2+y^2}$ is the **magnitude** of z, and θ is the **phase**, **angle**, or **argument** of z. Common notations for θ include $\angle z$ and $\arg z$. With this notation, we can write $z = |z|e^{j\arg z} = |z|\angle z$. For each $z \neq 0$, there are infinitely many possible values for $\arg z$, which all differ from each other by an integer multiple of 2π. For this reason, it is sometimes convenient to use the ***principal*** **angle** or ***principal*** **argument** of z, which is the *unique* value of $\theta \in (-\pi, \pi]$ for which $z = |z|e^{j\theta}$. The principal argument is denoted by $\operatorname{Arg} z$ *with an uppercase* "A". We show below that

$$\operatorname{Arg}(x+jy) = \begin{cases} \tan^{-1}(y/x), & x>0, & \text{right half-plane}, \\ \tan^{-1}(y/x)+\pi, & x<0, y\geq 0, & \text{upper left-half-plane}, \\ \tan^{-1}(y/x)-\pi, & x<0, y<0, & \text{lower left-half-plane}, \\ \pi/2, & x=0, y>0, & +j\text{-axis}, \\ -\pi/2, & x=0, y<0, & -j\text{-axis}, \\ \text{undefined}, & x=0, y=0, & \text{origin}, \end{cases} \tag{1}$$

and that

$$\operatorname{Arg}(x+jy) = \begin{cases} \pi/2 - \tan^{-1}(x/y), & y>0, \quad \text{upper half-plane,} \\ -\pi/2 - \tan^{-1}(x/y), & y<0, \quad \text{lower half-plane,} \\ 0, & y=0, x>0, \quad \text{positive real axis,} \\ \pi, & y=0, x<0, \quad \text{negative real axis,} \\ \text{undefined,} & x=0, y=0, \quad \text{origin.} \end{cases} \tag{2}$$

To obtain $\operatorname{Arg} z$ in MATLAB, use `angle(z)`, and note that `angle(0)` returns `0`.

Derivation of (1) and (2). Consider a complex number $z = x + jy$ in the first quadrant, as shown in Figure 1. When x and y are both positive, $\theta \in (0, \pi/2)$, and the the-

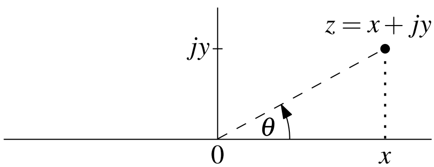

Figure 1. A complex number $z = x + jy$ in the first quadrant of the complex plane.

ory of right triangles tells us that $\tan\theta = y/x$. Similarly, $\tan(\pi/2 - \theta) = x/y$. Hence, we have two formulas to choose from: $\theta = \tan^{-1}(y/x)$ and $\theta = \pi/2 - \tan^{-1}(x/y)$. The first formula holds even for $x > 0$ and $y = 0$; i.e., for z on the positive real axis, the argument is zero. The second formula holds even for $x = 0$ and $y > 0$; i.e., for z on the $+j$-axis, the argument is $\pi/2$.

Now consider $z = x + jy$ in the second quadrant, as shown in Figure 2. Since

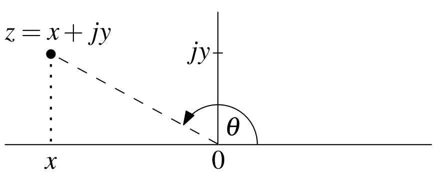

Figure 2. A complex number $z = x + jy$ in the second quadrant of the complex plane.

$\theta > \pi/2$, we apply the theory of right triangles to the the supplementary angle $\pi - \theta$. Since x is negative, the length of the horizontal side is $-x$. Thus, $\tan(\pi - \theta) = y/(-x)$, and, since $\tan^{-1}$ is an odd function, it follows that $\theta = \pi + \tan^{-1}(y/x)$. Since the complementary angle of $\pi - \theta$ is $\pi/2 - (\pi - \theta) = \theta - \pi/2$, we have $\tan(\theta - \pi/2) = -x/y$, and it follows that $\theta = \pi/2 - \tan^{-1}(x/y)$. ***Notice this formula is the same as the one derived in the first quadrant!*** Hence, for any z in the strict upper half-plane, its argument is $\pi/2 - \tan^{-1}(x/y)$.

For the third quadrant, it can similarly be shown that $\theta = \tan^{-1}(y/x) - \pi$ and $\theta = -\pi/2 - \tan^{-1}(x/y)$.

For the fourth quadrant, $\theta = \tan^{-1}(y/x)$ and $\theta = -\pi/2 - \tan^{-1}(x/y)$. Hence, the argument of any z in the strict lower half-plane is $-\pi/2 - \tan^{-1}(x/y)$, and the argument of any z in the strict right half-plane is $\tan^{-1}(y/x)$.

Functions of Frequency. Suppose $z(f) = x(f) + jy(f)$, where $x(f)$ and $y(f)$ are differentiable functions of f. Put $\theta(f) := \arg(x(f) + jy(f))$. Our goal is to compute $\theta'(f)$ for all f with $x(f)^2 + y(f)^2 > 0$. Note that differentiation removes any multiple of 2π that distinguishes different versions of arg. Hence, we can use any version that is convenient. There are four (overlapping) cases to consider: $x(f) > 0$, $x(f) < 0$, $y(f) > 0$, and $y(f) < 0$.

For $x(f) > 0$, we use the first formula in (1) to write

$$\tan(\theta(f)) = \tan(\tan^{-1}(y(f)/x(f)) = y(f)/x(f).$$

Differentiating $\tan(\theta(f)) = y(f)/x(f)$ with respect to f yields

$$\sec^2(\theta(f))\theta'(f) = \frac{x(f)y'(f) - y(f)x'(f)}{x(f)^2}.$$

Since $\sec^2 = 1 + \tan^2$, and since $\tan^2(\theta(f)) = y(f)^2/x(f)^2$, we can write

$$\big[1 + y(f)^2/x(f)^2\big]\theta'(f) = \frac{x(f)y'(f) - y(f)x'(f)}{x(f)^2}.$$

Solving for $\theta'(f)$, we obtain

$$\frac{\partial}{\partial f}\arg(x(f) + jy(f)) = \frac{x(f)y'(f) - y(f)x'(f)}{x(f)^2 + y(f)^2}. \tag{3}$$

For $y(f) < 0$, we use the second formula in (2), the identity $\tan(\pi/2 - \alpha) = 1/\tan\alpha$, and the fact that the tangent is odd. It follows that

$$\begin{aligned}\tan(\theta(f)) &= \tan(-\pi/2 - \tan^{-1}(x(f)/y(f)))\\ &= -\tan(\pi/2 + \tan^{-1}(x(f)/y(f)))\\ &= -1/\tan(-\tan^{-1}(x(f)/y(f))) = y(f)/x(f).\end{aligned}$$

Arguing as above, we see that (3) holds for $y(f) < 0$. It can similarly be shown that (3) holds for $y(f) > 0$.

To handle the case $x(f) < 0$, we cannot use Arg because it is discontinuous across the negative real axis. However, if we add 2π to the third formula in (1), the result will be identical to the second formula in (1). In other words, for $z = x + jy$ in the strict left half-plane, we can write $\arg(x+jy) = \tan^{-1}(y/x) + \pi$. Since the tangent function has period π, we have

$$\tan(\theta(f)) = \tan(\arg(x(f)+jy(f))) = \tan(\tan^{-1}(y(f)/x(f)) + \pi) = y(f)/x(f),$$

and (3) holds for $x(f) < 0$.

Putting everything together, we have

$$\boxed{\frac{\partial}{\partial f}\arg(x(f)+jy(f)) = \frac{x(f)y'(f) - y(f)x'(f)}{x(f)^2+y(f)^2}, \quad x(f)^2+y(f)^2 > 0.} \tag{4}$$