APPENDIX B
Mathematical Induction

Consider a statement about positive integers, for example

$$P(n): \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \tag{B.1}$$

where $n \geq 1$. Using (B.1), we see that P(1) is given by

$$\sum_{k=1}^{1} k = \frac{1(1+1)}{2},$$

which is obviously true. When P(n) is given by (B.1), P($n+1$) is given by

$$\begin{aligned} \sum_{k=1}^{n+1} k &= \frac{[n+1]([n+1]+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned} \tag{B.2}$$

For a general statement P(n), to prove that it is true for all positive integers using **mathematical induction** on n is to carry out the following two-step procedure:

1. Show that P(1) is true.
2. Fix an arbitrary $n \geq 1$ and show that if P(n) is true, then P($n+1$) is true; i.e., show that for $n \geq 1$,

$$P(n) \Rightarrow P(n+1).$$

We note that sometimes it is more convenient to prove that for all $n \geq 2$, we have $P(n-1) \Rightarrow P(n)$.

When P(n) is given by (B.1), we have already noted that P(1) is true. We now show that $P(n) \Rightarrow P(n+1)$. Suppose P(n) is true. We must show that P($n+1$) is true; i.e., we most show that (B.2) holds. So we write

$$\begin{aligned} \sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1), \quad \text{by the induction hypothesis } P(n), \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Example B.1. Show that if $h \geq -1$, then

$$\mathrm{P}(n): \quad (1+h)^n \geq 1+nh \tag{B.3}$$

holds for $n \geq 1$.

Solution. First, when $n = 1$, (B.3) becomes $(1+h) \geq 1+h$, which is obviously true. Suppose $(1+h)^n \geq 1+nh$. We must show that (B.3) holds for n replaced by $n+1$; i.e., we must show that

$$(1+h)^{n+1} \geq 1+(n+1)h.$$

To derive this, write

$$\begin{aligned}
(1+h)^{n+1} &= (1+h)^n(1+h) \\
&\geq (1+nh)(1+h), \quad \text{by the induction hypothesis (B.3),} \\
&= 1+nh+h+nh^2 \\
&\geq 1+(n+1)h, \qquad \text{since } nh^2 \geq 0.
\end{aligned} \tag{B.4}$$

Note that the inequality in (B.4) requires $(1+h) \geq 0$; i.e., $h \geq -1$.

Sometimes it is convenient to start the induction at $n = 0$.

Example B.2 (Division Algorithm). Let d be a given positive integer. Then

$$\mathrm{P}(n): \quad n = dq+r, \quad \text{for some } 0 \leq r < d \text{ and } q \geq 0,$$

holds for $n \geq 0$.

Solution. If $n = 0$, we can take $q = r = 0$. Now suppose the result is true for some $n \geq 0$. To show it is true for $n+1$, use the induction hypothesis to write

$$n = dq+r, \quad 0 \leq r < d.$$

Adding one to both sides of the above equation shows that

$$n+1 = dq+(r+1).$$

If $r+1 < d$, we have the desired representation of $n+1$. Otherwise $r+1 = d$, and the above display becomes $n+1 = d(q+1)$.

Problems

1. Show that for $n \geq 1$,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Show that for $n \geq 1$,

$$\sum_{k=1}^{n} (2k-1)^2 = \frac{n(2n-1)(2n+1)}{3}.$$

3. Recall the **binomial coefficients** $\binom{n}{k} := n!/[k!\,(n-k)!]$. Derive the **Leibniz rule**,

$$(xy)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} x^{(k)} y^{(n-k)}, \quad n \geq 0,$$

where $x^{(n)}$ is the nth derivative of x. In particular, $x^{(0)} = x$, $x^{(1)} = x'$, $x^{(2)} = x''$, etc. *Hint:* The easily verified identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \quad n \geq k \geq 1,$$

may be useful.