Basic Properties of Power Series

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Abstract

These notes provide a quick introduction (with proofs) to the basic properties of power series, including the exponential function and the fact that power series can be differentiated term by term. It is assumed that the reader is familiar with the following facts and concepts from analysis [4]:

- The triangle inequality [4, pp. 14–15, Theorem 1.13 and p. 23, Problem 13]: For complex a and b,

$$\big||a|-|b|\big| \leq |a+b| \leq |a|+|b|.$$

- The binomial theorem: For complex a and b,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad n = 0, 1, \ldots.$$

- If $z_n \to z$, then $|z_n| \to |z|$.
- A convergent sequence is bounded.
- If a series converges, its terms tend to zero and are therefore bounded.
- Cauchy sequences.
- A convergent sequence is Cauchy.
- The real and complex numbers are complete.
- Uniform convergence.

If you find this writeup useful, or if you find typos or mistakes, please let me know at gubner@engr.wisc.edu

1. Background Results from Analysis

Lemma 1. *For $n \geq 2$, $\sum_{k=1}^{n-1} k = n(n-1)/2$.*

Proof. Induction. □

Lemma 2. *If $h > 0$, then $(1+h)^n \geq 1 + nh$ for $n \geq 1$, and $(1+h)^n \geq 1 + nh + n(n-1)h^2/2$ for $n \geq 2$.*

Proof. These formulas are immediate from the binomial theorem, or they can be proved easily by induction on n. □

Lemma 3. *If $0 \leq t < 1$, then $t^n \to 0$, $nt^n \to 0$, and $(n-1)t^n \to 0$.*

Proof. The results are trivial for $t = 0$, so fix $0 < t < 1$. Then $h := (1/t) - 1 > 0$ and we can write $t = 1/(1+h)$. By Lemma 2,

$$t^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh} \leq \frac{1}{nh}$$

and

$$t^n = \frac{1}{(1+h)^n} \leq \frac{1}{1+nh+n(n-1)h^2/2} \leq \frac{2}{n(n-1)h^2}.$$

Now fix $\varepsilon > 0$. Then in the first case, the right-hand side is less than ε if $n > 1/(\varepsilon h)$. In the second case, we have $nt^n < \varepsilon$ if $n > 1 + 2/(\varepsilon h^2)$, and we have $(n-1)t^n < \varepsilon$ if $n > 2/(\varepsilon h^2)$. □

Remark. From $nt^n \to 0$, we can write $nt^{n-1} = (nt^n/t) \to 0/t = 0$.

Lemma 4. *If $t > 0$, then $t^{1/n} \to 1$.*

Proof. (Based on [4, pp. 57–58, Theorem 3.20(b)].) If $t = 1$, then $t^{1/n} = 1 \to 1$. If $t > 1$, then $t^{1/n} > 1$ and $x_n := t^{1/n} - 1 > 0$. Now use Lemma 2 to write

$$t = (1 + x_n)^n \geq 1 + nx_n.$$

Rearrange this as $x_n \leq (t-1)/n \to 0$. But $x_n \to 0$ implies $t^{1/n} \to 1$. If $0 < t < 1$, then $s := 1/t > 1$ and $s^{1/n} \to 1$ by the preceding argument. Hence, $t^{1/n} = 1/s^{1/n} \to 1$. □

Theorem 5 (Geometric Series). *If z is a complex number, then*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad |z| < 1.$$

Proof. For complex z, put

$$S_N(z) := \sum_{n=0}^{N-1} z^n. \tag{1}$$

Note that $S_N(1) = N$. For $z \neq 1$, write

$$S_N(z) - zS_N(z) = (1+z+\cdots+z^{N-1}) - (z+z^2+\cdots+z^N) = 1-z^N.$$

Rearranging yields

$$S_N(z) = \frac{1-z^N}{1-z}, \quad z \neq 1. \tag{2}$$

For $|z| < 1$, we must show that $S_N(z) \to 1/(1-z)$. This follows from (2) if we can show $z^N \to 0$. Since $|z| < 1$, we have $|z^N| = |z|^N \to 0$ by Lemma 3. $\square$

Theorem 6. *If z is a complex number, then*

$$\sum_{n=1}^{\infty} nz^{n-1} = \frac{1}{(1-z)^2}, \quad |z| < 1.$$

Proof. Differentiating (1) and (2) shows that

$$S_N'(z) = \sum_{n=1}^{N-1} nz^{n-1} = \frac{1-Nz^{N-1}+(N-1)z^N}{(1-z)^2}. \tag{3}$$

For $|z| < 1$, we have $N|z|^{N-1}$ and $(N-1)|z|^N$ both tending to zero by Lemma 3 and the Remark following it. $\square$

Theorem 7. *An absolutely convergent series converges.*

Proof. Let $w_1, w_2, \ldots$ be complex numbers, and put $\widetilde{W}_N := \sum_{n=0}^{N-1} |w_n|$ and $W_N := \sum_{n=0}^{N-1} w_n$. We must show that if $\widetilde{W}_N$ converges, then W_N also converges. For $M > N$, use the **triangle inequality** to write

$$|W_M - W_N| = \left| \sum_{n=N}^{M-1} w_n \right| \leq \sum_{n=N}^{M-1} |w_n| = \widetilde{W}_M - \widetilde{W}_N.$$

By hypothesis, $\widetilde{W}_N$ converges and is therefore Cauchy. The above inequality shows that W_N is also Cauchy. Since the complex numbers are complete, W_N converges. $\square$

Theorem 8 (Comparison Test). *Let $c_1, c_2, \ldots$ be a sequence of nonnegative numbers such that $\sum_{n=0}^{N-1} c_n$ converges. If $w_1, w_2, \ldots$ are complex numbers with $|w_n| \leq c_n$ holding for all sufficiently large n, then $\sum_{n=0}^{N-1} w_n$ and $\sum_{n=0}^{N-1} |w_n|$ converge.*

Proof. Put $C_N := \sum_{n=0}^{N-1} c_n$ and define $\widetilde{W}_N$ as in the proof of Theorem 7. Suppose $|w_n| \le c_n$ for all $n \ge N_0$. Then for $M > N \ge N_0$,

$$\widetilde{W}_M - \widetilde{W}_N = \sum_{n=N}^{M-1} |w_n| \le \sum_{n=N}^{M-1} c_n = C_M - C_N.$$

By hypothesis, C_N converges and is therefore Cauchy. The above inequality shows that $\widetilde{W}_N$ is also Cauchy and therefore converges. By Theorem 7, $\sum_{n=0}^{N-1} w_n$ also converges. ☐

Lemma 9. *If $w_1, w_2, \ldots$ are complex numbers, and if $\sum_{n=1}^{N} w_n$ converges, then*

$$\left|\sum_{n=1}^{\infty} w_n\right| \le \sum_{n=1}^{\infty} |w_n|.$$

Remark. The above right-hand side may be infinite, but if it is finite, then the series on the left converges by Theorem 7.

Proof. With the notation from proof of Theorem 7, let $W := \lim_{N\to\infty} W_N$ and $\widetilde{W} := \lim_{N\to\infty} \widetilde{W}_N$. We must show that $|W| \le \widetilde{W}$. By the triangle inequality, $\big||W_N| - |W|\big| \le |W_N - W| \to 0$, and so $|W_N| \to |W|$. We can now write

$$|W| = \lim_{N\to\infty} |W_N| = \lim_{N\to\infty} \left|\sum_{n=1}^{N} w_n\right| \le \lim_{N\to\infty} \sum_{n=1}^{N} |w_n| = \widetilde{W}.$$ ☐

Theorem 10 (Discrete Fubini). *Let w_{mn} be complex numbers with*

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} |w_{mn}|\right) < \infty.$$

Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} w_{mn}.$$

Proof. There are four important implications of the hypothesis.

- For each n, $\sum_{m=1}^{\infty} |w_{mn}| < \infty$. Hence, $\sum_{m=1}^{\infty} w_{mn}$ converges absolutely.
- For each m, $\sum_{n=1}^{\infty} |w_{mn}| < \infty$. Hence, $W_m^N := \sum_{n=1}^{N} w_{mn}$ converges absolutely to $W_m := \sum_{n=1}^{\infty} w_{mn}$.
- Using Lemma 9, $\sum_{m=1}^{\infty} |W_m| = \sum_{m=1}^{\infty} \left|\sum_{n=1}^{\infty} w_{mn}\right| \le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |w_{mn}| < \infty$. This shows that $\sum_{m=1}^{\infty} W_m$ converges absolutely.
- Given $\varepsilon > 0$, we can choose M so large that

$$\sum_{m=M+1}^{\infty}\left(\sum_{n=1}^{\infty} |w_{mn}|\right) < \varepsilon/2.$$

Using this notation, we must prove that

$$\sum_{m=1}^{\infty} W_m = \lim_{N\to\infty} \sum_{n=1}^{N} \sum_{m=1}^{\infty} w_{mn}.$$

For M as above, let N be so large that $|W_m - W_m^N| < \varepsilon/(2M)$ for $m = 1, \ldots, M$. Then

$$\begin{aligned}
\left| \sum_{m=1}^{\infty} W_m - \sum_{n=1}^{N} \sum_{m=1}^{\infty} w_{mn} \right| &= \left| \sum_{m=1}^{\infty} W_m - \sum_{m=1}^{\infty} \sum_{n=1}^{N} w_{mn} \right| \\
&= \left| \sum_{m=1}^{\infty} W_m - \sum_{m=1}^{\infty} W_m^N \right| \\
&\le \sum_{m=1}^{\infty} |W_m - W_m^N|, \quad \text{by Lemma 9,} \\
&= \sum_{m=1}^{M} |W_m - W_m^N| + \sum_{m=M+1}^{\infty} |W_m - W_m^N| \\
&\le \varepsilon/2 + \sum_{m=M+1}^{\infty} |W_m - W_m^N|.
\end{aligned}$$

Denoting this last sum by S_M, we have

$$S_M = \sum_{m=M+1}^{\infty} \left| \sum_{n=N+1}^{\infty} w_{mn} \right| \le \sum_{m=M+1}^{\infty} \sum_{n=N+1}^{\infty} |w_{mn}| \le \sum_{m=M+1}^{\infty} \sum_{n=1}^{\infty} |w_{mn}| < \varepsilon/2. \qquad \square$$

Theorem 11 (Uniform Cauchy Criterion). *Let $f_n(z)$ be a sequence of complex-valued functions defined on a subset E of the complex plane. If the f_n are uniformly Cauchy on E in the sense that for every $\varepsilon > 0$, there exists an N_0 such that for all $n, m \ge N_0$,*

$$|f_n(z) - f_m(z)| < \varepsilon, \quad \text{for all } z \in E, \tag{4}$$

then f_n converges uniformly on E.

Proof. If the f_n are uniformly Cauchy on E, then in particular for each $z \in E$, $f_n(z)$ is a Cauchy sequence and therefore converges to some complex number, which we denote by $f(z)$. Hence, $f_n(z) \to f(z)$ for each $z \in E$. Given $\varepsilon > 0$, let N_0 be such that for all $n, m \ge N_0$, (4) holds. We claim that for all $n \ge N_0$,

$$|f_n(z) - f(z)| < 2\varepsilon, \quad \text{for all } z \in E.$$

Fix any $z \in E$. Since $f_n(z) \to f(z)$, for all sufficiently large m, $|f_m(z) - f(z)| < \varepsilon$. Fix any such m that is also greater than N_0. Then for $n \ge N_0$,

$$|f_n(z) - f(z)| \le |f_n(z) - f_m(z)| + |f_m(z) - f(z)| < \varepsilon + \varepsilon = 2\varepsilon. \qquad \square$$

2. Basic Properties

Put

$$A_N(z) := \sum_{n=0}^{N-1} a_n z^n \quad \text{and} \quad \widetilde{A}_N(z) := \sum_{n=0}^{N-1} |a_n z^n|,$$

where the a_n are arbitrary complex numbers. When $A_N(z)$ is known to converge, we denote the limit by $A(z)$, and when $\widetilde{A}_N(z)$ is known to converge, we denote its limit by $\widetilde{A}(z)$. This notation allows us to emphasize that the value of an infinite series is the limit of the partial sums.

Associated with the above series, put

$$\kappa := \limsup_{n\to\infty} |a_n|^{1/n}.$$

The **radius of convergence** is $r := 1/\kappa$, where $1/0$ is taken as ∞, and $1/\infty$ is taken as zero.[1]

Example 12. If $a_n = 1/n!$, show that $\kappa = 0$.

Solution. Let $\varepsilon > 0$ be given. Let $N+1 \geq 1/\varepsilon$. Then for $n > N$,

$$n! = 1\cdot 2\cdots N(N+1)\cdots n \geq N!\left(\frac{1}{\varepsilon}\right)^{n-N} = \frac{N!\,\varepsilon^N}{\varepsilon^n},$$

or

$$\left(\frac{1}{n!}\right)^{1/n} \leq \varepsilon\left(\frac{1}{N!\,\varepsilon^N}\right)^{1/n}.$$

By Lemma 4, for sufficiently large n, $\big(1/(N!\,\varepsilon^N)\big) < 2$, which implies $(1/n!)^{1/n} < 2\varepsilon$. Since ε was arbitrary, $(1/n!)^{1/n} \to 0$.

Theorem 13 (Radius of Convergence). *If $|z| < r$, then $A_N(z)$ converges absolutely, and if $|z| > r$, then $A_N(z)$ does not converge. Furthermore, if $0 < \rho < r$, then for some finite constant M, we have $|a_n| < M/\rho^n$ for all n.*

Proof. The conditions $|z| < r$ and $|z| > r$ are equivalent to $|z|\kappa < 1$ and $|z|\kappa > 1$. It is convenient to put

$$\theta(z) := |z|\kappa = \limsup_{n\to\infty} |a_n z^n|^{1/n}.$$

If $\theta(z) < 1$, choose $\varepsilon > 0$ so small that $\theta(z) + \varepsilon < 1$. The definition of lim sup implies that for all sufficiently large n we have $|a_n z^n|^{1/n} < \theta(z) + \varepsilon < 1$. Equivalently,

[1] Hence, κ is the **curvature** of a circle of radius r.

$|a_n z^n| \leq [\theta(z)+\varepsilon]^n$. This last quantity is the typical term in a convergent geometric series, and so by the Comparison Test, $A_N(z)$ converges absolutely.

Now suppose $\theta(z) > 1$. To obtain a contradiction, suppose $A_N(z)$ converges. Then $|a_n z^n| \to 0$. However, using the definition of lim sup in the definition of $\theta(z)$ implies that there is a subsequence $|a_{n_k} z^{n_k}|^{1/n_k} \to \theta(z) > 1$. Hence, there are infinitely many values of n with $|a_n z^n|^{1/n} \geq 1$; i.e., $|a_n z^n| \geq 1$. This contradicts $|a_n z^n| \to 0$.

If we take $z = \rho$ in the first paragraph of the proof, then for all sufficiently large n, say for $n > N$, we have $|a_n \rho^n|^{1/n} < 1$, which implies $|a_n \rho^n| < 1$, or $|a_n| < 1/\rho^n$. By taking $M \geq 1$ and $M > \max_{1\leq n\leq N} |a_n|\rho^n$, we have $|a_n| < M/\rho^n$ for all n. □

Theorem 14 (Uniform Convergence). *If $A_N(z)$ has radius of convergence r, then $A_N(z)$ converges uniformly on any disk of radius strictly less than r.*

Proof. Fix any $0 \leq r_1 < r$. Since $r_1 \geq 0$, there is some complex z_1 with $|z_1| = r_1$. Hence, $r_1 = |z_1| < r$. Then $A_N(z_1)$ converges absolutely. Equivalently, $\widetilde{A}_N(z_1)$ converges and is therefore Cauchy. We now apply the Uniform Cauchy Criterion Theorem 11 as follows. For $|z| \leq r_1 = |z_1|$ and $M > N$, write

$$|A_N(z) - A_M(z)| = \left| \sum_{n=N}^{M-1} a_n z^n \right| \leq \sum_{n=N}^{M-1} |a_n|\,|z|^n \leq \sum_{n=N}^{M-1} |a_n|\,|z_1|^n = \widetilde{A}_M(z_1) - \widetilde{A}_N(z_1).$$

Since $\widetilde{A}_N(z_1)$ is Cauchy, the inequality shows that $A_N(z)$ is uniformly Cauchy. □

Example 15. We now see that $\exp(z) := \sum_{n=0}^{\infty} z^n/n!$ converges absolutely and uniformly for all complex z.

Theorem 16. *For complex z and w,* $\exp(z+w) = \exp(z)\exp(w)$.

Proof. Using the binomial theorem, write

$$\exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}.$$

We know that the middle expression above is a finite complex number for all complex numbers z and w. In particular this is true when z and w are replaced by $|z|$ and $|w|$, respectively. Hence,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} |z|^k |w|^{n-k} < \infty.$$

By the Discrete Fubini Theorem 10, the following calculations are justified. Let $u(t)$ denote the **unit-step function**, $u(t) := 1$ for $t \geq 0$ and $u(t) := 0$ for $t < 0$. Then

$$\exp(z+w) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \binom{n}{k} z^k w^{n-k} u(n-k)$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{n=0}^{\infty} \frac{w^{n-k}}{(n-k)!} u(n-k)$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{n=k}^{\infty} \frac{w^{n-k}}{(n-k)!}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \exp(z)\exp(w). \qquad \square$$

Lemma 17 (Multiplication of Power Series). *Let $A_N(z)$ and $B_N(z)$ each have a positive radius of convergence, and let r denote the smaller of the radii. Then for $|z| < r$, $C(z) := A(z)B(z)$ is equal to $\sum_{n=0}^{\infty} c_n z^n$, where*

$$c_n := \sum_{k=0}^{n} a_k b_{n-k}, \quad n = 0, 1, \ldots.$$

Proof. For $|z| < r$, we can write

$$C(z) = A(z)B(z) = \left(\sum_{k=0}^{\infty} a_k z^k\right) B(z) = \sum_{k=0}^{\infty} a_k B(z) z^k = \sum_{k=0}^{\infty} a_k \left(\sum_{\ell=0}^{\infty} b_\ell z^\ell\right) z^k$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_k b_\ell z^{\ell+k}.$$

We can similarly write

$$\left(\sum_{k=0}^{\infty} |a_k z^k|\right)\left(\sum_{\ell=0}^{\infty} |b_\ell z^\ell|\right) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} |a_k|\,|b_\ell|\,|z|^{\ell+k},$$

where the left-hand side is finite because power series are absolutely convergent. Hence, we can apply Theorem 10 when we need it shortly. Recalling the unit-step function u introduced earlier, write

$$C(z) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_k b_\ell z^{\ell+k} = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_k b_{n-k} z^n = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} a_k b_{n-k} z^n u(n-k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k b_{n-k} z^n u(n-k)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} z^n \qquad \square$$

Theorem 18 (Differentiation of Power Series). *Suppose $A_N(z)$ has radius of convergence r. Then for all $|z| < r$,*

$$\frac{d}{dz}\sum_{n=0}^{\infty} a_n z^n = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Proof. Put $B_N(z) := \sum_{n=1}^{N-1} n a_n z^{n-1}$. We first argue that $B_N(z)$ converges for $|z| < r$. From the proof of Theorem 13, we can write $|n a_n z^{n-1}| = (n/|z|)|a_n z^n| \leq (n/|z|)[\theta(z)+\varepsilon]^n$. This last quantity is the typical term in a convergent series (multiply the result of Theorem 6 by z to see this). By the Comparison Test, $B_N(z)$ converges absolutely, and we denote this limit by $B(z)$.

Fix any $|z_0| < r$ and choose ρ with $|z_0| < \rho < r$. We must show that

$$\frac{A(z)-A(z_0)}{z-z_0} - B(z_0)$$

tends to zero as $z \to z_0$. Since $A(z)$, $A(z_0)$, and $B(z_0)$ are given by convergent series,

$$\begin{aligned}
A(z)-A(z_0)-B(z_0)(z-z_0) &= \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n z_0^n - \left[\sum_{n=1}^{\infty} n a_n z_0^{n-1}\right](z-z_0) \\
&= \sum_{n=1}^{\infty} a_n (z^n - z_0^n) - \sum_{n=1}^{\infty} n a_n z_0^{n-1}(z-z_0) \\
&= \sum_{n=1}^{\infty} a_n \left[z^n - z_0^n - n z_0^{n-1}(z-z_0)\right] \\
&= \sum_{n=2}^{\infty} a_n \left[z^n - z_0^n - n z_0^{n-1}(z-z_0)\right].
\end{aligned}$$

Now write

$$\begin{aligned}
z^n - z_0^n - n z_0^{n-1}(z-z_0) &= z^n - z_0^n - n z_0^{n-1} z + n z_0^n \\
&= z^n + (n-1) z_0^n - n z_0^{n-1} z \\
&= z^n \left[1 + (n-1)(z_0/z)^n - n(z_0/z)^{n-1}\right] \\
&= z^n (z-z_0)^2 \sum_{k=1}^{n-1} k (z_0/z)^{k-1}, \quad \text{by (3)}, \\
&= (z-z_0)^2 \sum_{k=1}^{n-1} k z_0^{k-1} z^{n-k+1}.
\end{aligned}$$

Since $|z_0| < \rho$, for z close to z_0, we have $|z| < \rho$ as well, and so

$$|A(z)-A(z_0)-B(z)(z-z_0)| = |z-z_0|^2 \left|\sum_{n=2}^{\infty} a_n \left[\sum_{k=1}^{n-1} k z_0^{k-1} z^{n-k+1}\right]\right|$$

$$\le |z-z_0|^2 \sum_{n=2}^{\infty} |a_n| \sum_{k=1}^{n-1} k|z_0|^{k-1}|z|^{n-k+1}, \quad \text{by Lemma 9,}$$

$$\le |z-z_0|^2 \sum_{n=2}^{\infty} |a_n|\rho^n \sum_{k=1}^{n-1} k, \quad \text{since } |z|, |z_0| < \rho,$$

$$= |z-z_0|^2 \sum_{n=2}^{\infty} |a_n|\rho^n n(n-1)/2, \quad \text{by Lemma 1.}$$

If we can show this last sum is finite, then we can divide both sides by $|z-z_0|$ and see that

$$\left| \frac{A(z)-A(z_0)}{z-z_0} - B(z_0) \right| \to 0 \quad \text{as } z \to z_0.$$

To show that this last sum is finite, use the proof of Theorem 13 with $z = \rho$ to write

$$|a_n|\rho^n n(n-1)/2 \le [\theta(\rho)+\varepsilon]^n n(n-1)/2.$$

The right-hand side is the typical term in a convergent series (differentiate (3) and let $N \to \infty$), and so the desired result follows by the Comparison Test. □

References

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