***Example* 13.23** (substitution law). If $\mathsf{E}[|w(X,Y)|] < \infty$, show that

$$\mathsf{E}[w(X,Y)|Y=y] \;=\; \mathsf{E}[w(X,y)|Y=y],$$

where it is understood that the right-hand side is computed using the conditional density or conditional pmf of X given Y.[4]

Solution. Let us denote the above right-hand side by $\widehat{g}(y)$. In the density case, $\widehat{g}(y) = \int_{-\infty}^{\infty} w(x,y) f_{X|Y}(x|y)\,dx$. We must show that for every bounded function g,

$$\mathsf{E}[w(X,Y)g(Y)] = \mathsf{E}[\widehat{g}(Y)g(Y)].$$

Write

$$\begin{aligned}
\mathsf{E}[\widehat{g}(Y)g(Y)] &= \int_{-\infty}^{\infty} \widehat{g}(y)g(y)f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} w(x,y)f_{X|Y}(x|y)\,dx\right]g(y)f_Y(y)\,dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)g(y)f_{XY}(x,y)\,dx\,dy \\
&= \mathsf{E}[w(X,Y)g(Y)].
\end{aligned}$$

The pmf case is similar.

***Note* 4.** The general form of the substitution law is

$$\mathsf{E}[w(X,Y)|Y=y] = \int w(x,y)\,P(dx,y),$$

whenever X possesses a **regular conditional probability** given Y, denoted here by P. The derivation is very similar to that given in Example 13.23, but uses the result of Problem 18.20 in Billingsley. To say that P is a regular conditional probability for X given Y means that $\mathsf{P}(X \in B|Y=y) = P(B,y)$, where P is such that for fixed y, $P(B,y)$ is a measure as a function of B, and for fixed B, $P(B,y)$ is a measurable function of y. Fortunately, whenever X takes values in $\mathbb{R}$, in $\mathbb{R}^n$, or even in $\mathbb{R}^\infty$, a regular conditional probability always exists (see Breiman or Shiryayev). The reason that $\mathsf{E}[w(X,y)|Y=y]$ must be understood as the above right-hand side is discussed on p. 80 of Breiman and in the papers by Bahadur and Bickel and by Proschan and Presnell.

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