Alternating Projections onto Convex Sets

Based on
References


**Properties of projections**

**Lemma.** Let $C$ be a convex subset of an inner product space. Then $\hat{x} \in C$ satisfies

$$\|x - \hat{x}\| \leq \|x - y\| \text{ for all } y \in C \quad (1)$$

$$\iff \Re \langle x - \hat{x}, y - \hat{x} \rangle \leq 0 \text{ for all } y \in C. \quad (2)$$

**Proof.** $\iff$: Write

$$\|x - y\|^2 = \|(x - \hat{x}) + (\hat{x} - y)\|^2$$

$$= \|x - \hat{x}\|^2 + 2 \Re \langle x - \hat{x}, \hat{x} - y \rangle + \|\hat{x} - y\|^2$$

$$\geq \|x - \hat{x}\|^2$$

$\Rightarrow$: Suppose (1) holds. Then for any $y \in C$ and any $0 < \lambda < 1$, we can write

$$\|x - \hat{x}\|^2 \leq \|x - \{\hat{x} + \lambda(y - \hat{x})\}\|^2$$

$$= \|(x - \hat{x}) - \lambda(y - \hat{x})\|^2$$

$$= \|x - \hat{x}\|^2 - 2 \lambda \Re \langle x - \hat{x}, y - \hat{x} \rangle + \lambda^2 \|y - \hat{x}\|^2$$

Rearranging and using the fact that $\lambda > 0$, $\Re \langle x - \hat{x}, y - \hat{x} \rangle \leq \lambda \|y - \hat{x}\|^2/2$.

Since $\lambda > 0$ can be arbitrarily small, (2) follows
Suppose \( C \) is a closed convex subset of a Hilbert space \( \mathcal{X} \). By the Projection Theorem, for every \( x \in \mathcal{X} \), there exists an \( \hat{x} \in C \) satisfying (1). The projection operator onto \( C \) is defined by

\[
P_x = \hat{x}.
\]

Note: If \( C \) is not a subspace, \( P \) is not linear!!!

**Proposition.** If \( C \) is a closed convex subset of a Hilbert space \( \mathcal{X} \), then for every \( x_1, x_2 \in \mathcal{X} \),

\[
\| \hat{x}_1 - \hat{x}_2 \|^2 \leq \text{Re} \left< x_1 - x_2, \hat{x}_1 - \hat{x}_2 \right> \tag{3}
\]

and

\[
\| P_{x_1} - P_{x_2} \| \leq \| x_1 - x_2 \|. \tag{4}
\]

**Proof.** First note that (3) \( \Rightarrow \) (4) by Cauchy-Schwarz.

To prove (3), we first observe that for all \( y \in C \),

\[
\text{Re} \left< x_1 - \hat{x}_1, y - \hat{x}_1 \right> \leq 0 \tag{5}
\]

and

\[
\text{Re} \left< x_2 - \hat{x}_2, y - \hat{x}_2 \right> \leq 0 \tag{6}
\]

In (5) set \( y = \hat{x}_2 \) and in (6) take \( y = \hat{x}_1 \). Then

(5) becomes
\[
\text{Re} \left\langle x_1 - \hat{x}_1, \hat{x}_2 - \hat{x}_1 \right\rangle \leq 0, \tag{7}
\]

and (6) becomes
\[
\text{Re} \left\langle \hat{x}_2 - x_2, \hat{x}_2 - \hat{x}_1 \right\rangle \leq 0. \tag{8}
\]

Adding (7) and (8) yields
\[
\text{Re} \left\langle (x_1 - x_2) - (\hat{x}_1 - \hat{x}_2), \hat{x}_2 - \hat{x}_1 \right\rangle \leq 0. \tag{9}
\]

Rearranging (9) yields (3).

**Definition.** A mapping \( T : X \to Y \), where \( X \) and \( Y \) are normed vector spaces, is nonexpansive if

\[
\| T x_1 - T x_2 \| \leq \| x_1 - x_2 \|, \quad \forall x_1, x_2 \in X.
\]

By (4), we see that projection operators are nonexpansive. Note also that nonexpansive operators are uniformly continuous.
Relaxed Projection Operators

Let $C$ be a closed convex subset of a Hilbert space $X$. Let $P$ be the corresponding projection operator.

Define

$$Tx = \lambda(Px) + (1-\lambda)x$$

$$= x + \lambda(\hat{x} - x)$$

For $0 \leq \lambda \leq 1$, $Tx$ is a convex combination of $x$ and $\hat{x}$.

**Example:**

![Diagram](image)

$Tx$ for $\lambda > 1$

**Lemma** For $0 < \lambda < 2$, $Tx = \lambda(Px) + (1-\lambda)x$ is nonexpansive.

**Proof.** For $0 < \lambda \leq 1$, write

$$||Tx - Ty|| = ||\lambda(Px - Py) + (1-\lambda)(x - y)||$$

$$\leq \lambda ||Px - Py|| + (1-\lambda)||x - y||,$$ \quad \text{triangle ineq.}

$$\leq \lambda||x - y|| + (1-\lambda)||x - y||,$$ \quad \text{since $P$ is nonexp.}

$$= ||x - y||.$$

For $1 < \lambda < 2$, we proceed as follows. Write

$$||Tx - Ty||^2 = \lambda^2||Px - Py||^2 + (1-\lambda)^2||x - y||^2$$

$$+ 2(1-\lambda)^2 \Re \langle x - y, Px - Py \rangle$$

$$< 0$$
By (3), $\Re \langle x-y, P_k-P_y \rangle \geq \|P_k-P_y\|^2$. Hence,

\[
\|Tx-Ty\|^2 \leq \lambda^2 \|P_x-P_y\|^2 + (1-\lambda)^2 \|x-y\|^2 \\
+ 2\lambda(1-\lambda) \|P_x-P_y\|^2 \\
= \lambda(2-\lambda) \|P_x-P_y\|^2 + (1-\lambda)^2 \|x-y\|^2 \\
\leq \lambda(2-\lambda) \|x-y\|^2 + (1-\lambda)^2 \|x-y\|^2 \\
= \{2\lambda - \lambda^2 + 1 - 2\lambda + \lambda^2 + \lambda^3 \|x-y\|^2 \\
= \|x-y\|^2.
\]

Suppose $C_1, \ldots, C_m$ are closed convex subsets of a Hilbert space $X$. Let $P_1, \ldots, P_m$ be the corresponding projection operators. Let $0 < \lambda_i < 2$, $i = 1, \ldots, m$. Let

\[ T_i = I + \lambda_i (P_i - I), \]

where $I$ is the identity operator on $X$; i.e. $Ix = x$, $x \in X$. By the last lemma, each $T_i$ is a nonexpansive, relaxed projection operator. Define

\[ T = T_m T_{m-1} \cdots T_2 T_1, \]

In other words,

\[ T(x) = T_m (T_{m-1} (\cdots (T_2 (T_1 (x)) \cdots)). \]
Since each $T_i$ is nonexpansive, so is $T=T_m \cdots T_1$.

Define

$$C_0 \overset{\circ}{=} \bigcap_{i=1}^{m} C_i$$

and

$$\mathcal{T} \overset{\Delta}{=} \{ x \in X : T x = x \}.$$ 

We call $\mathcal{T}$ the set of fixed points of $T$.

**Lemma.** $C_0 \subset \mathcal{T}$.

**Proof.** Suppose $x \in C_0$. Since $C_0 \subset C_1$, $x \in C_1$. Since $x \in C_1$, $P_i x = x$. Since $P_i x = x$, $T_i x = x$. So,

$$T x = T_m(\cdots (T_i(T_i x))\cdots)$$

$$= T_m(\cdots (T_2 x)\cdots).$$

Since $x \in C_0 \subset C_2$, $T_2 x = x$. Continuing in this way,

$T x = x$. Thus $C_0 \subset \mathcal{T}$. 

We show later that in fact $C_0 = \mathcal{T}$ if $C_0 \neq \emptyset$.

We also show later that $\mathcal{T}$ has the following property.

**Definition.** A mapping $T : X \rightarrow X$ is asymptotically regular if

$$T^n x - T^{n+1} x \rightarrow 0.$$
The following theorem shows that if $T = T_m \cdots T_1$, and if $T$ has a fixed point, then for all $x \in X$, $T^n x$ converges to a fixed point of $T$; that is, $T^n x$ converges to an element of $T = C_0$. In general, the fixed point to which $T^n x$ converges will not be the projection of $x$ onto $C_0$.

**Theorem.** (Opial) [Finite-dimensional version].
Let $C$ be a closed convex subset of a finite-dimensional inner product space $X$. (In our application to alternating projections, we take $C = X$.) Let $T : C \to C$ be any nonexpansive, asymptotically regular mapping. Let $T$ denote the set of fixed points of $T$. Assume $T \neq \emptyset$. Then for every $x \in C$, $T^n x$ converges to a point in $T$.

**Proof:** For every $y \in T$, $Ty = y$. Since $T$ is nonexpansive, for $y \in T$,

$$\|T^n x - y\| = \|T^{n+1} x - Ty\| \leq \|T^n x - y\|. \quad (10)$$

Thus,

$$\|T^n x - y\| \leq \|x - y\|, \quad \text{all } n \geq 1.$$  

In particular,

$$\|T^n x\| \leq \|x - y\| + \|y\|. \quad (11)$$

Since $T = C_0$, $T$ has a fixed point $\Rightarrow C_0 \neq \emptyset$. 

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and hence \( \{T^n x\} \) is a bounded sequence in a finite-dimensional inner product space. Therefore, there is a subsequence \( \{T^{n_k} x\}_{k=1}^{\infty} \) and a point \( y_0 \in X \) with \( T^{n_k} x \to y_0 \) as \( k \to \infty \). Since \( T^{n_k} x \in C \), and \( C \) is closed, \( y_0 \in C \).

We claim \( y_0 \) is a fixed point of \( T \); i.e., \( y_0 \in T \).

Since \( T \) is asymptotically regular,
\[
T^{n_k} x - T^{n_k+1} x \to 0.
\]
Rewrite this as
\[
(I-T)(T^{n_k} x) \to 0.
\]
It is readily verified that \( I-T \) is continuous; hence,
\[
(I-T)(T^{n_k} x) \to (I-T)(y_0).
\]
Thus, \( (I-T)(y_0) = 0 \), and so \( Ty_0 = y_0 \).

We now show that \( T^n x \to y_0 \). Since \( y_0 \in T \), we can take \( y=y_0 \) in (10). This shows that the sequence of real numbers \( \{\|T^n x - y_0\|\} \) is nonincreasing and bounded below. By the monotonic sequence property,
\[
\lim_{n \to \infty} \|T^n x - y_0\| \text{ exists.}
\]
Now, \( \{\|T^{n_k} x - y_0\|\}_{k=1}^{\infty} \) is a subsequence of \( \{\|T^n x - y_0\|\} \).

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The assumption \( T \neq \emptyset \) is crucial, since \( T \neq \emptyset \) implies there is a \( y \) satisfying (11).
Hence,
\[ \lim_{n \to \infty} \|T^n x - y_0\| = \lim_{k \to \infty} \|T^k x - y_0\| = 0 \text{ since } T^k x \to y_0. \]

But now \( \lim_{n \to \infty} \|T^n x - y_0\| = 0 \) implies \( T^n x \to y_0 \).

**Proposition.** Let \( 0 < \lambda_i < 2 \), \( i = 1, \ldots, m \). Assume \( C_0 = \bigcap_{i=1}^m C_i \neq \emptyset \).

Then, letting \( T_i = I + \lambda_i (P_i - I) \) and \( T = T_m \cdots T_1 \),

\[ \|x - T_i x\|^2 \leq \frac{\lambda_i}{2 - \lambda_i} \left( \|x - y\|^2 - \|T_i x - y\|^2 \right), \quad \forall y \in C_i, \quad (12) \]

and

\[ \|x - Tx\|^2 \leq b_m 2^{m-1} \left( \|x - y\|^2 - \|Tx - y\|^2 \right), \quad \forall y \in C_0, \quad (13) \]

where \( b_m = \max_{1 \leq i \leq m} \left\{ \lambda_i / (2 - \lambda_i) \right\} \).

Further, \( C_0 = \bigcap_{i=1}^m C_i \).

**Proof.** We first verify (12). We begin with the observation that

\[ \text{Re} \left< x - P_i x, y - P_i x \right> - \|x - P_i x\|^2 \]

\[ = \text{Re} \left< x - P_i x, (y - P_i x) - (x - P_i x) \right> \]

\[ = \text{Re} \left< x - P_i x, y - x \right> \]

\[ = \text{Re} \left< x - y, P_i x - x \right>, \quad 2 "-" \text{ signs } + \text{ a complex conjugate.} \]

(14)
We next observe that

\[
\| T_i x - y \| ^2 = \| x - y \| ^2 + 2 \lambda_i \Re \langle x - y, P_i x - x \rangle + \lambda_i^2 \| P_i x - x \| ^2
\]

\[
= \| x - y \| ^2 + 2 \lambda_i \left( \Re \langle x - P_i x, y - P_i x \rangle - \| x - P_i x \| ^2 \right)
+ \lambda_i^2 \| P_i x - x \| ^2, \quad \text{by (14)},
\]

\[
\leq \| x - y \| ^2 - \lambda_i (2 - \lambda_i) \| x - P_i x \| ^2 + 2 \lambda_i \Re \langle x - P_i x, y - P_i x \rangle
\]

\[
\leq \| x - y \| ^2 - \lambda_i (2 - \lambda_i) \| x - P_i x \| ^2
\]

\[
= \| x - y \| ^2 - \frac{2 - \lambda_i}{\lambda_i} \| x - T_i x \| ^2, \quad \text{(15)}.
\]

where the last step follows because \( \| x - T_i x \| ^2 = \lambda_i^2 \| x - P_i x \| ^2 \).

Rearranging (15) yields (12).

The next step is to prove (13) by induction on \( m \).

The case \( m = 1 \) is covered by (12). For \( m \geq 2 \), define \( K = T_{m-1} \cdots T_1 \). Then \( T = T_m K \). The induction hypothesis is expressed in terms of \( K \) by

\[
\| x - K x \| ^2 \leq \sum_{i=1}^{m-1} 2^{m-2} \| x - y \| ^2 - \| K x - y \| ^2, \quad y \in \bigcap_{i=1}^{m-1} C_i.
\]

Before continuing, we recall the inequality \((u + v)^2 \leq 2(u^2 + v^2)\).
This is derived by writing $0 \leq (u-v)^2 = u^2 - 2uv + v^2$, which implies $2uv \leq u^2 + v^2$. From this, $(u+v)^2 = u^2 + 2uv + v^2 \leq u^2 + (u^2 + v^2) + v^2$.

Fix $y \in \bigcap_{i=1}^{m} C_i$. Then

$$
\|x - Tx\|^2 = \|x - Kx + Kx - Tx\|^2 \\
\leq \left( \|x - Kx\|^2 + \|Kx - Tx\|^2 \right), \text{ by the } \Delta \text{ ineq.} \\
\leq 2 \left( \|x - Kx\|^2 + \|Kx - Tx\|^2 \right) \\
\leq 2 \left( \|x - Kx\|^2 + 2^{m-2} \|Kx - Tx\|^2 \right).
$$

(17)

Now,

$$
\|Kx - Tx\|^2 = \|(Kx) - Tm(Kx)\|^2 \\
\leq \frac{\lambda_m}{2 - \lambda_m} \left( \|(Kx) - y\|^2 - \|Tm(Kx) - y\|^2 \right),
$$

by applying (12) with $i = m$, and $Kx$ in place of $x$

$$
\leq b_m \left( \|Kx - y\|^2 - \|T_x - y\|^2 \right).
$$

(18)

Substituting (16) and (18) into (17) yields

$$
\|x - T_x\|^2 \leq 2 \left( b_{m-1} 2^{m-2} \left[ \|x - y\|^2 - \|Kx - y\|^2 \right] \\
+ 2^{m-2} b_m \left[ \|Kx - y\|^2 - \|T_x - y\|^2 \right] \right)
$$

$$
\leq 2 \left( b_m 2^{m-2} \left[ \|x - y\|^2 - \|T_x - y\|^2 \right] \right),
$$

which is exactly (13).
We conclude the proof of the Proposition by showing that \( C_0 = T \). Recall that on p. 9 we showed that \( C_0 \subset T \). To show \( T = C_0 \), we proceed as follows. Fix \( x \in T \). If we can show that \( T_i x = x \) for each \( i \), then \( P_i x = x \) for each \( i \). Then \( P_i x = x \) implies \( x \in C_i \), and thus \( x \in \bigcap_{i=1}^{m} C_i = C_0 \).

We first show \( T_i x = x \). Since we assumed \( C_0 \neq \emptyset \), there is a \( y \in C_0 \). For such \( y \), \( T_y = y \) (since \( C_0 \subset T \)). Write

\[
\|x - y\| = \|T x - T y\| \leq \|T_i x - T_i y\|,
\]

since \( T_m \cdots T_2 \) is nonexpansive,

\[
= \|T_i x - y\|, \text{ since } T y = y \text{ for } y \in C_0,
\]

\[
= \|T x - T_i y\|,
\]

\[
\leq \|x - y\|, \text{ since } T_i \text{ is nonexpansive}.
\]

Thus, \( \|x - y\| = \|T_i x - y\| \). Applying this in (12) yields \( T x = x \).

To prove \( T_j x = x \) in general, we use induction on \( j \). Fix \( x \in T \), and suppose \( T_i x = x \) for \( i = 1, \ldots, j-1 \). Then \( x \in \bigcap_{i=1}^{j-1} C_i \).

Write

\[
\|x - y\| = \|T x - T y\| = \|T_{m \cdots T_{j+1}} T_j x - T_{m \cdots T_{j+1}} T_{j-1} y\|
\]

since \( x \in \bigcap_{i=1}^{j-1} C_i \Rightarrow T_{j-1} \cdots T_i x = x \), by the induction hypothesis.

\[
\leq \|T_j x - T_j y\|, \text{ since } T_m \cdots T_{j+1} \text{ is nonexpansive}.
\]

\[
= \|T_j x - y\|, \text{ since } y \in C_0
\]

\[
= \|T_j x - T_j y\|
\]

\[
\leq \|x - y\|
\]

Thus, \( \|x - y\| = \|T_j x - y\| \). By (12), \( T_j x = x \). This completes the induction. \( \square \)
Corollary. \( T = T_1 \cdots T_n \) is asymptotically regular.

Proof. We must show that \( T^n x - T^{n+1} x \to 0 \). This will follow immediately if we show that
\[
\sum_{n=0}^{\infty} \| T^n x - T^{n+1} x \|^2 < \infty.
\] (19)

To this end, replace \( x \) with \( T^n x \) in (13). Then
\[
\| T^n x - T^{n+1} x \|^2 \leq b_m 2^{m-1} \left( \| T^n x - y \|^2 - \| T^{n+1} x - y \|^2 \right).
\]

For \( N \geq 2 \),
\[
\sum_{n=0}^{N-1} \| T^n x - T^{n+1} x \|^2 \leq b_m 2^{m-1} \left( \| x - y \|^2 - \| T^N x - y \|^2 \right) \leq b_m 2^{m-1} \| x - y \|^2.
\]

Letting \( N \to \infty \),
\[
\sum_{n=0}^{\infty} \| T^n x - T^{n+1} x \|^2 \leq b_m 2^{m-1} \| x - y \|^2 < \infty.
\]

Remark. Any mapping satisfying (19) is called a reasonable wanderer.
Relaxed Projections onto Subspaces

We consider the special case in which every \( C_i \) is a closed subspace of a Hilbert space. We begin with a preliminary Lemma.

**Lemma.** Let \( M \) be a closed subspace of a Hilbert space \( \mathbb{H} \), and let \( P \) denote the corresponding projection operator. Then (i) \( P \) is linear, (ii) \( P \) is bounded, and (iii) \( P \) is self adjoint.

**Proof.** (i) follows by the orthogonality principle and by the uniqueness of the projection. (ii) follows by writing
\[
\|x\|^2 = \|x - \hat{x} + \hat{x}\|^2 = \|x - \hat{x}\|^2 + \|\hat{x}\|^2 \geq \|\hat{x}\|^2 = \|Px\|^2.
\]
To prove (iii), write
\[
\langle Px, y \rangle = \langle \hat{x}, y \rangle = \langle \hat{x}, \hat{y} + (y - \hat{y}) \rangle = \langle \hat{x}, \hat{y} \rangle = \langle \hat{x} + (x - \hat{x}), \hat{y} \rangle = \langle x, \hat{y} \rangle = \langle x, Py \rangle.
\]

The next theorem, which does not rely on Opial's Theorem, shows that if each \( C_i \) is a closed subspace, then if \( T = T_m \cdots T_1 \), \( T^n x \) converges to \( P_0 x = \) the projection of \( x \) onto \( C_0 = \bigcap_{i=1}^m C_i. \)
Theorem. Let $C_0 = \bigcap_{i=1}^{m} C_i$, where each $C_i$ is a closed subspace of a Hilbert space $X$. Let $P_i$, $i=0,\ldots,m$, denote the corresponding projection operator. For $i=1,\ldots,m$, let $0 < \lambda_i < 2$, and let $T_i = I + \lambda_i (P_i - I)$. Set $T = T_m \cdots T_1$. Then for every $x \in X$, $T^nx \to P_0 x$.

Proof. We begin by pointing out that since each $C_i$ is a subspace, so is $C_0$. Hence $C_0$ contains the zero vector. Thus, we can apply the Proposition on p. 12. In particular, we obtain $C_0 = \mathcal{T}$.

Next, since $P_i$ is linear, bounded, and self-adjoint, so is $T_i$. It follows that $T = T_m \cdots T_1$ is linear, bounded, and satisfies $T^* = T_1 \cdots T_m$.

Hence, observe that our proof that $C_0 = \mathcal{T}$ also shows that $x \in C_0 \iff T^*x = x$.

Now, by the Corollary on p. 16, $T$ is asymptotically regular. Hence, since $T^n$ is linear,

$$T^n[(I-T)(x)] = T^n x - T^{n+1} x \to 0.$$  \hspace{1cm} (20)

Thus, if $y = (I-T)(x)$ for some $x$, $T^ny \to 0$. In other words, $T^ny \to 0$ for all $y \in \text{range of } I-T$. Since $I-T$ is a bounded linear operator on a Hilbert space,

$$\text{range of } I-T = \overline{\text{ker}(I-T)^*} = \overline{\text{ker}(I-T^*)} = \overline{\text{ker}(I-T^*)}.$$
Since \( x \in C_0 \iff T^*x = x \iff (I - T^*)(x) = 0 \iff x \in \ker(I - T^*) \), we see that \( C_0 = \ker(I - T^*) \). Thus

\[
\mathcal{X} = C_0 \oplus C_0^\perp
\]

\[
= C_0 \oplus \overline{\ker(I - T^*)}
\]

\[
= C_0 \oplus \mathcal{R}(I - T).
\]

Fix any \( x \in \mathcal{X} \). Then \( x \) can be uniquely written as

\[
x = P_0x + y,
\]

where \( y \in \mathcal{R}(I - T) \). Clearly, \( T(x) = P_0x + Ty \). In general, \( T^n x = P_0x + T^n y \). We will be finished if we can show that \( T^n y \to 0 \). To this end, recall that since \( y \in \mathcal{R}(I - T) \), there is a sequence \( y_k \in \mathcal{R}(I - T) \) with \( y_k \to y \). Let \( \epsilon > 0 \) be given. Choose \( k \) large enough that \( \|y - y_k\| < \epsilon/2 \). Since \( y_k \in \mathcal{R}(I - T) \), there is some \( x_k \) with \( y_k = (I - T)(x_k) \). Thus, for large \( n \),

\[
\|T^n y\| \leq \|T^n y_k - T^n y_k\| + \|T^n y_k\| \text{ Write since } y_k \in \mathcal{R}(I - T), \text{ let } n \text{ be large enough that } \|T^n y_k\| < \epsilon/2 \text{. Then }
\]

\[
\|T^n y\| \leq \|T^n y - T^n y_k\| + \|T^n y_k\| \leq \|y - y_k\| + \|T^n y_k\| \leq \|y - y_k\| + \epsilon/2 \leq \epsilon/2 + \epsilon/2 = \epsilon.
\]