The Gamma Function and Stirling's Formula

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Abstract

Starting with Euler's integral definition of the gamma function, we state and prove the Bohr–Mollerup Theorem, which gives Euler's limit formula for the gamma function. We then discuss two independent topics. The first is upper and lower bounds on the gamma function, which lead to Stirling's Formula. The second is the Euler–Mascheroni Constant and the digamma function.

If you find this writeup useful, or if you find typos or mistakes, please let me know at John.Gubner@wisc.edu

1. Summary

1.1. Euler's Integral Definition

The **gamma function**,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0,$$

has the following three properties:

(*i*) $\Gamma(1) = 1$.
(*ii*) $\Gamma(x+1) = x\Gamma(x)$. Use integration by parts.
(*iii*) $\ln\Gamma(x)$ is convex. By the **Hölder inequality**,

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \leq \Gamma(x)^{1/p}\Gamma(y)^{1/q},$$

where $0 < p < \infty$ with $1/p + 1/q = 1$ [6, p. 192].

1.2. Euler's Limit Formula

The **Bohr–Mollerup Theorem**, which we prove in Section 2, says that $\Gamma(x)$ is the only positive function on $(0,\infty)$ with these three properties. More specifically, the theorem says that if f is any positive function on $(0,\infty)$ satisfying the three properties $f(1)=1$, $f(x+1)=xf(x)$, and $\varphi(x):=\ln f(x)$ is convex, then

$$\varphi(x)=\lim_{n\to\infty}\ln\frac{n!\,n^x}{x(x+1)\cdots(x+n)}. \tag{1}$$

Since Γ is such a function,

$$\ln\Gamma(x)=\lim_{n\to\infty}\ln\frac{n!\,n^x}{x(x+1)\cdots(x+n)}. \tag{2}$$

Applying the exponential to both sides and interchanging it with the limit on the right[1] yields

$$\boxed{\Gamma(x)=\lim_{n\to\infty}\frac{n!\,n^x}{x(x+1)\cdots(x+n)}.}$$

1.3. Stirling's Formula[2]

In Section 3, we use the Bohr–Mollerup Theorem to show that there is a positive constant C such that

$$\boxed{Cx^{x-1/2}e^{-x}\le\Gamma(x)\le Cx^{x-1/2}e^{-x}e^{1/(12x)}.} \tag{3}$$

From the left-hand inequality,

$$C\le\lim_{x\to\infty}\frac{\Gamma(x)}{x^{x-1/2}e^{-x}},$$

and from the right-hand inequality,

$$\lim_{x\to\infty}\frac{\Gamma(x)}{x^{x-1/2}e^{-x}}\le\lim_{x\to\infty}Ce^{1/(12x)}=C.$$

Hence,

$$C=\lim_{x\to\infty}\frac{\Gamma(x)}{x^{x-1/2}e^{-x}}.$$

Since $\Gamma(n)=(n-1)!$, if we let $x=n$ in (3) and multiply the inequality by n, we obtain

$$\boxed{Cn^{n+1/2}e^{-n}\le n!\le Cn^{n+1/2}e^{-n}e^{1/(12n)}.}$$

[1] This is justified because the exponential is a continuous function [6, p. 86, Th. 4.6].

[2] The material in Subsection 1.3 and Section 3 can be read independently of the material in Subsections 1.4–1.5 and Section 4.

Hence, we can also write

$$C = \lim_{n\to\infty} \frac{n!}{n^{n+1/2}e^{-n}}.$$

It can be shown $C = \sqrt{2\pi}$; e.g., [2]–[4], [6, pp. 194–195].

It is shown in Section 3.2 that

$$\Gamma(x) \le 3x^x, \quad x \ge 1/2.$$

1.4. The Euler–Mascheroni Constant

Observe that the quotient in (2) satisfies

$$\begin{aligned}\frac{n!\,n^x}{x(x+1)\cdots(x+n)} &= \frac{e^{-x/1}\cdots e^{-x/n}\cdot e^{x/1}\cdots e^{x/n}n^x}{x(1+x)(1+x/2)\cdots(1+x/n)}\\ &= e^{x(\ln n-1-1/2-\cdots-1/n)}\frac{1}{x}\frac{e^x}{1+x}\frac{e^{x/2}}{1+x/2}\cdots\frac{e^{x/n}}{1+x/n}.\end{aligned}$$

Taking the logarithm of both sides yields

$$\ln\frac{n!\,n^x}{x(x+1)\cdots(x+n)} = x\left(\ln n - \sum_{k=1}^{n}\frac{1}{k}\right) - \ln x + \sum_{k=1}^{n}\left[\frac{x}{k} - \ln\left(1+\frac{x}{k}\right)\right]. \tag{4}$$

The limit of the left-hand side exists and is equal to $\ln\Gamma(x)$ by the Bohr–Mollerup Theorem, and we show in Section 4 that

$$\boxed{\sum_{k=1}^{n}\frac{1}{k} - \ln n \to \gamma,}$$

where γ is the **Euler–Mascheroni constant**. Hence, the sum on the right in (4) also converges, and so

$$\ln\Gamma(x) = -x\gamma - \ln x + \sum_{k=1}^{\infty}\left[\frac{x}{k} - \ln\left(1+\frac{x}{k}\right)\right].$$

1.5. The Digamma Function

Differentiating the preceding expression, assuming we can interchange the differentiation and the infinite sum, yields the **digamma function**[3]

$$\psi(x) := \frac{d}{dx}\ln\Gamma(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left[\frac{1}{k} - \frac{1/k}{1+x/k}\right]$$

[3] Of course $\psi(x) = \Gamma'(x)/\Gamma(x)$ as well.

$$= -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left[\frac{1}{k} - \frac{1}{k+x}\right]$$
$$= -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\frac{x}{k(k+x)}.$$

This sum converges uniformly on any finite interval of the form $(0,r]$. Hence, by [6, p. 152, Th. 7.17], we are justified in interchanging differentiation and the infinite sum.

An analogous uniform-convergence argument shows that

$$\psi'(x) = \frac{1}{x^2} + \sum_{k=1}^{\infty}\frac{1}{(k+x)^2}.$$

Since ψ' is positive, ψ is increasing; since ψ' is decreasing, ψ is concave. These two properties of ψ are illustrated in Figure 1. Of course, since we already knew that $\ln\Gamma(x)$ is convex, its derivative, ψ, is increasing.

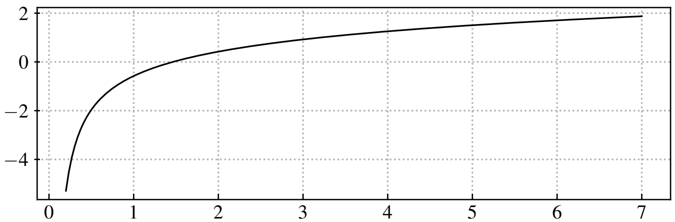

Figure 1. The digamma function ψ is increasing and concave.

2. Proof of the Bohr–Mollerup Theorem

Let f any positive function on $(0,\infty)$ satisfying the three properties $f(1)=1$, $f(x+1)=xf(x)$, and $\varphi(x) := \ln f(x)$ is convex, then (1) holds.

Before proving (1) for all $x>0$, we make a few observations. First, when $x=1$, the quotient in (1) reduces to $n/(n+1)\to 1$, and so the right-hand side has the correct value zero. Second, since $f(x+1)=xf(x)$, we have $\varphi(x+1)=\ln x+\varphi(x)$. So it would be a good sign if the limit in (1) also has this property. To see that it does, observe that if we replace x with $x+1$ in the quotient in (1), we can write

$$\frac{n!\,n^{(x+1)}}{(x+1)(x+2)\cdots(x+1+n)} = \frac{n!\,n^x}{(x+1)(x+2)\cdots(x+n)}\cdot\frac{n}{x+1+n}$$
$$= \frac{n!\,n^x}{x(x+1)(x+2)\cdots(x+n)}\cdot\frac{xn}{x+1+n},$$

where the rightmost factor tends to x. Hence,

$$\lim_{n\to\infty} \ln \frac{n!\,n^{(x+1)}}{(x+1)(x+2)\cdots(x+1+n)} = \lim_{n\to\infty} \ln \frac{n!\,n^x}{x(x+1)(x+2)\cdots(x+n)} + \ln x,$$

if the limit on the right exists (which is yet to be proved). If this limit is equal to $\varphi(x)$, then the preceding equation says that the limit on the left is equal to $\varphi(x)+\ln x$, which is equal to $\varphi(x+1)$ since we assumed $f(x+1) = xf(x)$ and $\varphi(x) := \ln f(x)$.

As a consequence of the foregoing observation, it suffices to prove that (1) holds for $0 < x < 1$. (We already showed that it holds for $x = 1$.) To do this, we apply induction to the property $\varphi(x+1) = \ln x + \varphi(x)$. For example,

$$\begin{aligned}\varphi(x+3) &= \ln(x+2) + \varphi(x+2)\\ &= \ln(x+2) + \ln(x+1) + \varphi(x+1)\\ &= \ln(x+2) + \ln(x+1) + \ln x + \varphi(x).\end{aligned}$$

In general,

$$\varphi(x+n) = \ln[x(x+1)\cdots(x+(n-1))] + \varphi(x).$$

In particular, with $x = 1$, we have $\varphi(n+1) = \ln n!$.

Following Rudin [6, p. 193], we use the convexity of φ with the three intervals[4]

$$[n, n+1], \quad [n+1, n+1+x], \quad \text{and} \quad [n+1, n+2].$$

Since $0 < x < 1$, we have $n < n+1 < n+1+x < n+2$, and the difference quotients of a convex function satisfy

$$\frac{\varphi(n+1)-\varphi(n)}{(n+1)-n} \le \frac{\varphi(n+1+x)-\varphi(n+1)}{(n+1+x)-(n+1)} \le \frac{\varphi(n+2)-\varphi(n+1)}{(n+2)-(n+1)}.$$

Using the properties of φ, we simplify the preceding inequality to obtain

$$\ln n \le \frac{\ln[x(x+1)\cdots(x+n)] + \varphi(x) - \ln n!}{x} \le \ln(n+1).$$

Multiplying through by x and then subtracting $x \ln n$, we have

$$0 \le \varphi(x) - \ln\left[\frac{n!\,n^x}{x(x+1)\cdots(x+n)}\right] \le x\ln(1+1/n).$$

Since the right-hand side tends to zero as $n \to \infty$, (1) holds for $0 < x < 1$. □

[4] According to Artin [1, p. vi], the use of convexity in this proof is due to H. Bohr and J. Mollerup in vol. III of their 1922 complex-analysis textbook.

3. Bounds on the Gamma Function

Following [1, Ch. 3], we select $\mu(x)$ so that

$$f(x) := x^{x-1/2}e^{-x}e^{\mu(x)}, \quad x > 0,$$

satisfies $f(x+1) = xf(x)$ and is log convex. Then by the Bohr–Mollerup Theorem, f will be proportional to the gamma function. Introducing upper and lower bounds on $\mu(x)$ will lead to bounds on the gamma function.

To solve $f(x+1) = xf(x)$, write

$$\begin{aligned} \frac{f(x+1)}{f(x)} &= \frac{(x+1)^{x+1/2}e^{-(x+1)}e^{\mu(x+1)}}{x^{x-1/2}e^{-x}e^{\mu(x)}} \\ &= \frac{(x+1)^x(x+1)^{1/2}e^{-x}e^{-1}e^{\mu(x+1)}}{x^x x^{1/2}x^{-1}e^{-x}e^{\mu(x)}} \\ &= (1+1/x)^{x+1/2}xe^{-1}e^{\mu(x+1)-\mu(x)}. \end{aligned}$$

To make this equal to x, we need

$$e^{\mu(x)-\mu(x+1)} = (1+1/x)^{(x+1/2)}e^{-1},$$

or

$$\begin{aligned} \mu(x) - \mu(x+1) &= (x+1/2)\ln(1+1/x) - 1 \\ &=: g(x). \end{aligned}$$

If the sum

$$\mu(x) := \sum_{n=0}^{\infty} g(x+n),$$

converges, then $\mu(x) - \mu(x+1)$ is a telescoping sum that is trivially equal to $g(x)$.

We show in Section 3.1 that g is convex, strictly decreasing, positive, and satisfies

$$0 < g(x) \le \frac{1}{12x(x+1)} = \frac{1}{12}\left[\frac{1}{x} - \frac{1}{x+1}\right].$$

To show that the series for $\mu(x)$ converges, we use the fact that the terms are positive to write

$$0 \le \mu(x) = \sum_{n=0}^{\infty} g(x+n) \le \frac{1}{12}\sum_{n=0}^{\infty}\left[\frac{1}{x+n} - \frac{1}{x+n+1}\right] = \frac{1}{12x}$$

due to the telescoping nature of the sum.

If we put $C := e^{1-\mu(1)}$, then $Cf(x)$ satisfies all the hypotheses of the Bohr–Mollerup Theorem. Hence,

$$\Gamma(x) = Cf(x) = Cx^{x-1/2}e^{-x}e^{\mu(x)}.$$

Since $0 \le \mu(x) \le 1/(12x)$, (3) follows.

3.1. Properties of the g Function

Using the geometric series, we can write the derivative of $\ln(1-x)$ as

$$\frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Since the series converges uniformly on any closed subinterval of $(-1,1)$, we can integrate term by term [6, p. 150, Th. 7.16] to get

$$\ln(1-x) = \int_0^x \frac{-1}{1-t}\,dt = -\sum_{n=0}^{\infty}\int_0^x t^n\,dt = -\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} = -\sum_{n=1}^{\infty}\frac{x^n}{n}.$$

Replacing x with $-x$ yields

$$\ln(1+x) = -\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}.$$

Following [1, p. 21], observe that

$$\tfrac{1}{2}\ln\frac{1+x}{1-x} = \tfrac{1}{2}\sum_{n=1}^{\infty}\frac{x^n}{n}[1-(-1)^n] = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1};$$

replace x with $1/(2x+1)$, which lies in $(0,1)$ for $x > 0$, to obtain

$$\tfrac{1}{2}\ln\Big(1+\frac{1}{x}\Big) = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2x+1)^{2n-1}};$$

multiply both sides by $(2x+1)$ so that

$$(x+1/2)\ln(1+1/x) = \sum_{n=1}^{\infty}\frac{1}{(2n-1)(2x+1)^{2n-2}},$$

and subtract one from both sides to yield

$$\begin{aligned} g(x) := (x+1/2)\ln(1+1/x) - 1 &= \sum_{n=2}^{\infty}\frac{1}{(2n-1)(2x+1)^{2n-2}} \\ &= \sum_{n=1}^{\infty}\frac{1}{(2n+1)(2x+1)^{2n}}. \end{aligned}$$

Using the definition of g, it is easy to check that $g''(x) > 0$, which implies g is convex. From the series, we see that g is positive and strictly decreasing. Furthermore, since $2n+1 \geq 3$ for $n \geq 1$, we can upper bound the series by writing

$$0 < g(x) \leq \sum_{n=1}^{\infty} \frac{1}{3(2x+1)^{2n}} = \frac{1}{3} \sum_{n=1}^{\infty} \left(\frac{1}{(2x+1)^2} \right)^n$$
$$= \frac{1}{12x(x+1)},$$

where the last step follows by the geometric series formula.

3.2. A Simpler Upper Bound

The logarithm of the upper bound in (3) can be written as

$$x \ln x + \ln C - (1/2) \ln x - x + \tfrac{1}{12x}.$$

Since the derivative of the last three terms,

$$-\tfrac{1}{2x} - 1 - \tfrac{1}{12x^2},$$

is negative, the sum of these terms themselves is maximized at the leftmost value of $x > 0$ under consideration. For example, if we restrict $x \geq x_0 = 1/2$, we have

$$\ln \Gamma(x) \leq x \ln x + \ln C - (1/2) \ln(1/2) - (1/2) + 1/6, \quad x \geq 1/2,$$

or

$$\Gamma(x) \leq \exp\big[\ln C - (1/2) \ln(1/2) - (1/2) + 1/6\big] x^x, \quad x \geq 1/2.$$

Since $C = \sqrt{2\pi}$, the exponential factor is less than 2.54003736144178. In particular, $\Gamma(x) \leq 3x^x$ for $x \geq 1/2$ is used in Vershynin [8, p. 26].

4. Convergence to the Euler–Mascheroni Constant

To establish the convergence of

$$\gamma_n := \sum_{k=1}^{n} \frac{1}{k} - \ln n,$$

it suffices to show that γ_n is nonincreasing and bounded below [6, p. 55, Th. 3.14]. The key is the simple pair of inequalities for $k \geq 1$,

$$\frac{1}{k+1} \leq \int_k^{k+1} \frac{1}{t}\, dt \leq \frac{1}{k},$$

which are equivalent to

$$\frac{1}{k+1} \le \ln\frac{k+1}{k} \le \frac{1}{k}. \tag{5}$$

From the left-hand inequality with k replaced with $k-1$, we obtain

$$\frac{1}{k} - \ln\frac{k}{k-1} \le 0$$

for $k \ge 2$. Since

$$\ln n = \sum_{k=2}^{n}[\ln k - \ln(k-1)] = \sum_{k=2}^{n}\ln\frac{k}{k-1},$$

we can write

$$\gamma_n = 1 + \sum_{k=2}^{n}\frac{1}{k} - \sum_{k=2}^{n}\ln\frac{k}{k-1} = 1 + \sum_{k=2}^{n}\left(\frac{1}{k} - \ln\frac{k}{k-1}\right).$$

Since the terms in this last sum are negative, γ_n is decreasing. To lower bound the sum, we introduce the smaller sequence

$$\begin{aligned}\widetilde{\gamma}_n := \gamma_n - \ln\frac{n+1}{n} &= \left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) - \ln\frac{n+1}{n} \\ &= \sum_{k=1}^{n}\frac{1}{k} - \ln(n+1) \\ &= \sum_{k=1}^{n}\frac{1}{k} - [\ln(k+1) - \ln k] \\ &= \sum_{k=1}^{n}\left(\frac{1}{k} - \ln\frac{k+1}{k}\right).\end{aligned}$$

The terms of this sum are positive on account of the right-hand inequality in (5). Hence, $\gamma_n \ge \widetilde{\gamma}_n \ge 0$. Since γ_n is monotonically decreasing and bounded below, it converges to some limit γ. Furthermore, since

$$\gamma_n - \widetilde{\gamma}_n = \ln\frac{n+1}{n} = \ln(1+1/n) \to \ln 1 = 0,$$

it follows that $\widetilde{\gamma}_n$ converges to the same limit as γ_n[5]

[5] Write

$$\lim_{n\to\infty}\widetilde{\gamma}_n = \lim_{n\to\infty}\left[(\widetilde{\gamma}_n - \gamma_n) + \gamma_n\right] = \lim_{n\to\infty}(\widetilde{\gamma}_n - \gamma_n) + \lim_{n\to\infty}\gamma_n = 0 + \gamma = \gamma,$$

where the second equal sign is justified by the fact that the limit of the sum on the left is equal to the sum of the limits on the right because the individual limits exist.

On account of the fact that $\widetilde{\gamma}_n$ is increasing and γ_n is decreasing, we have that $\widetilde{\gamma}_n \le \gamma \le \gamma_n$. Hence, the average of $\widetilde{\gamma}_n$ and γ_n will be closer to γ than either one of them. Consider the approximation

$$\frac{\widetilde{\gamma}_n + \gamma_n}{2} = \sum_{k=1}^{n} \frac{1}{k} - \frac{\ln(n+1) + \ln n}{2}.$$

This formula on the right with $n = 15$ yields 0.578, while the true value is $\gamma = 0.577$ to three significant digits. In contrast, $\gamma_{15} = 0.610$ and $\widetilde{\gamma}_{15} = \gamma_{15} - \ln(16/15) = 0.546$.

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